1. Generating functionals and asymptotic series

The generating functional $Z[J]$ of a free real scalar field theory is given by

$$Z_0[J] = \int d\varphi \, \exp\{-\tfrac{1}{2}\int d^4x\, \varphi(x)(\Delta + m^2)\varphi(x) + \int d^4x\, \varphi(x)J(x)\} \tag{1}$$

with

$$\langle \varphi(x_1)\cdots\varphi(x_{2n})\rangle = \sum_\sigma \prod_{i=1}^n G(x_{\sigma(2i-1)}, x_{\sigma(2i)}). \tag{2}$$

Here $G(x)$ is the Feynman propagator of $(\Delta + m^2)$. The sum $\sum_\sigma$ in (2) denotes the sum over all permutations σ of $(1, ..., 2n)$ with $\sigma(2i-1) < \sigma(2i)$ and $\sigma(2i-1) < \sigma(2i+1)$. Connected Green functions have the generating functional

$$W[J] = \ln Z[J], \qquad \text{with} \qquad \langle \phi(x_1)\cdots\phi(x_n)\rangle_{\rm c,J=0} := \prod_{i=1}^n \frac{\delta}{\delta J(x_i)} W[J]|_{J=0}. \tag{3}$$

W is called the Schwinger functional.

a) Convince yourself that (2) is valid and show that we have

$$\langle \varphi(x_1)\cdots\varphi(x_n)\rangle_{\rm c} = G_F(x_1, x_2; m)\, \delta_{n2}. \tag{4}$$

b) Consider the function $Z(\lambda)$ with coupling $\lambda > 0$:

$$Z(\lambda) := \int_{-\infty}^{\infty} d\varphi \, \exp\left[-\frac{1}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4\right] . \tag{5}$$

Compute the coefficients Z_n within the perturbative expansion in powers of λ,

$$Z = \sum_{n=0}^{\infty} Z_n \lambda^n \, , \qquad \text{tip:} \quad \int_0^\infty dt\, e^{-t} t^x \; = \; \Gamma(x+1) \, . \tag{6}$$

What is the radius of convergence in an expansion about $\lambda = 0$?

c) The remainder R_N of the partial sum of order N can be estimated by

$$R_N = |Z(\lambda) - \sum_0^N Z_n \lambda^n| \leq \lambda^{N+1} |Z_{N+1}| \quad . \tag{7}$$

The proof of (7) is attached below. Use the Stirling formula

$$\Gamma(x \to \infty) \to x^{x-\frac{1}{2}}\, e^{-x} \sqrt{2\pi} \tag{8}$$

to estimate $\lambda^n Z_n$ for large n. Estimate the order $N = N_{min}$, in which the remainder of the above partial sum is minimised.

Proof of (7): The above estimate follows with

$$\begin{aligned} R_N &= \int d\varphi\, e^{-\frac{1}{2}\varphi^2}\, |e^{-\frac{1}{4}\lambda\varphi^4} - \sum_{n=0}^{N} \tfrac{1}{n!}(-\lambda)^n(\tfrac{1}{4}\varphi^4)^n| \\ &\leq \int d\varphi\, e^{-\frac{1}{2}\varphi^2}\, \tfrac{1}{(N+1)!}\lambda^{N+1}(\tfrac{1}{4}\varphi^4)^{N+1} = \lambda^{N+1}|Z_{N+1}|. \end{aligned}$$

It follows from exercise 1c, that R_N approaches a minimum for $N_{\min} \approx (4\lambda)^{-1}$.

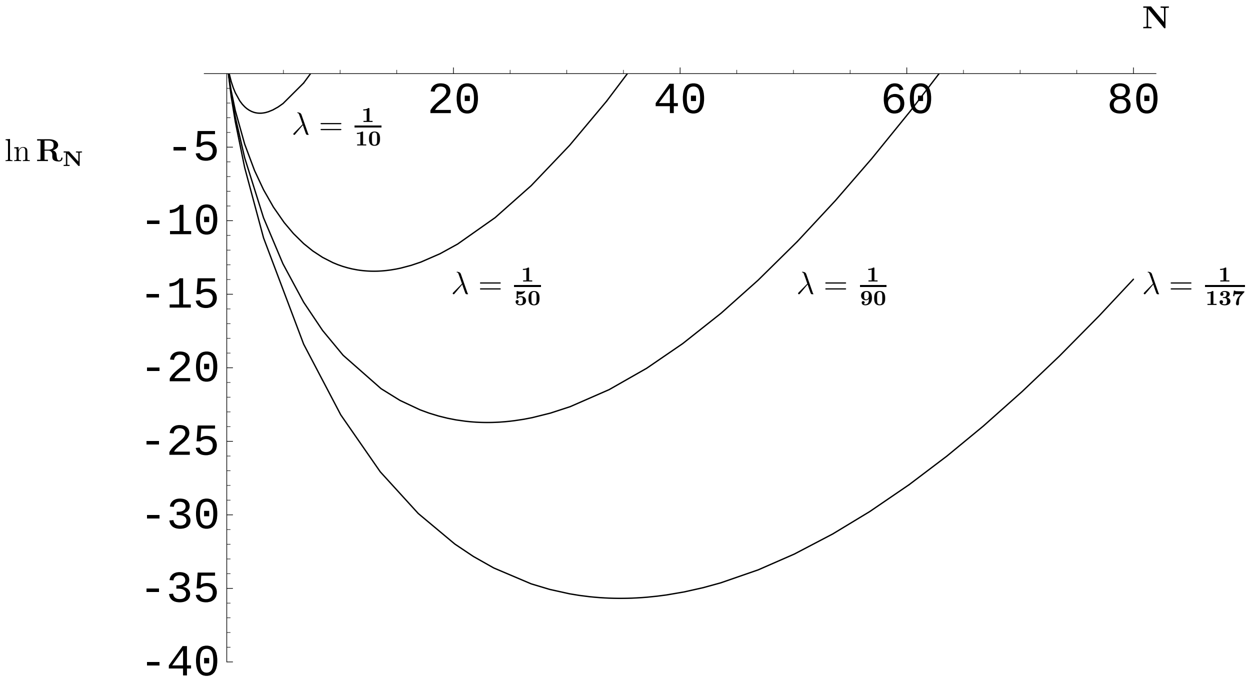

Figure 1: $\ln R_N$ for $\lambda = \frac{1}{10}, \frac{1}{50}, \frac{1}{90}, \frac{1}{137}$ as a function of N.

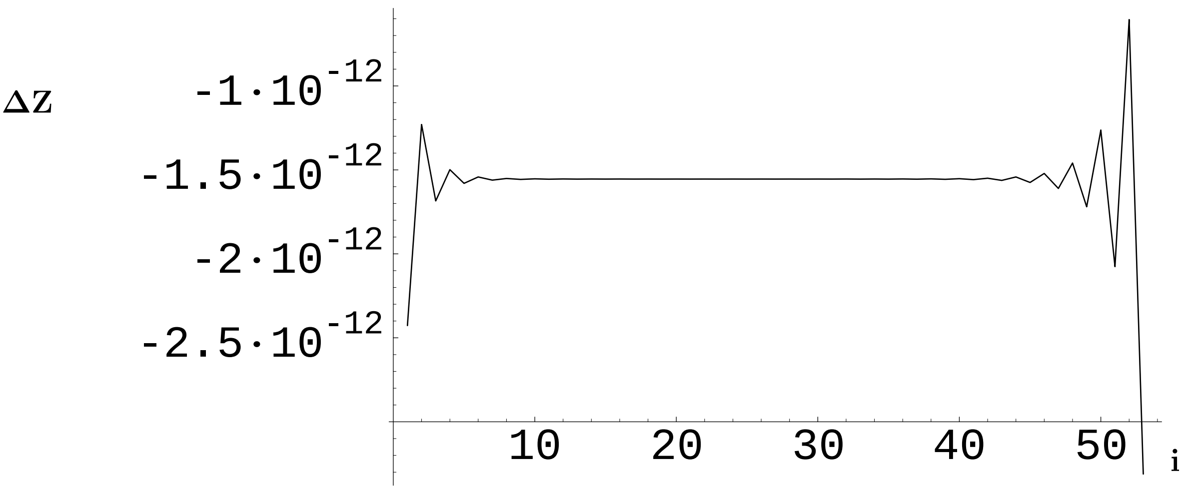

Figure 2: Graph of relative deviation $\Delta Z = (Z(\lambda) - \sum_{n=1}^{11+i} Z_n\lambda^n)/Z(\lambda)$ für $\lambda = 1/137$. The optimum is at about $N \approx 34$ ($i \approx 23$)