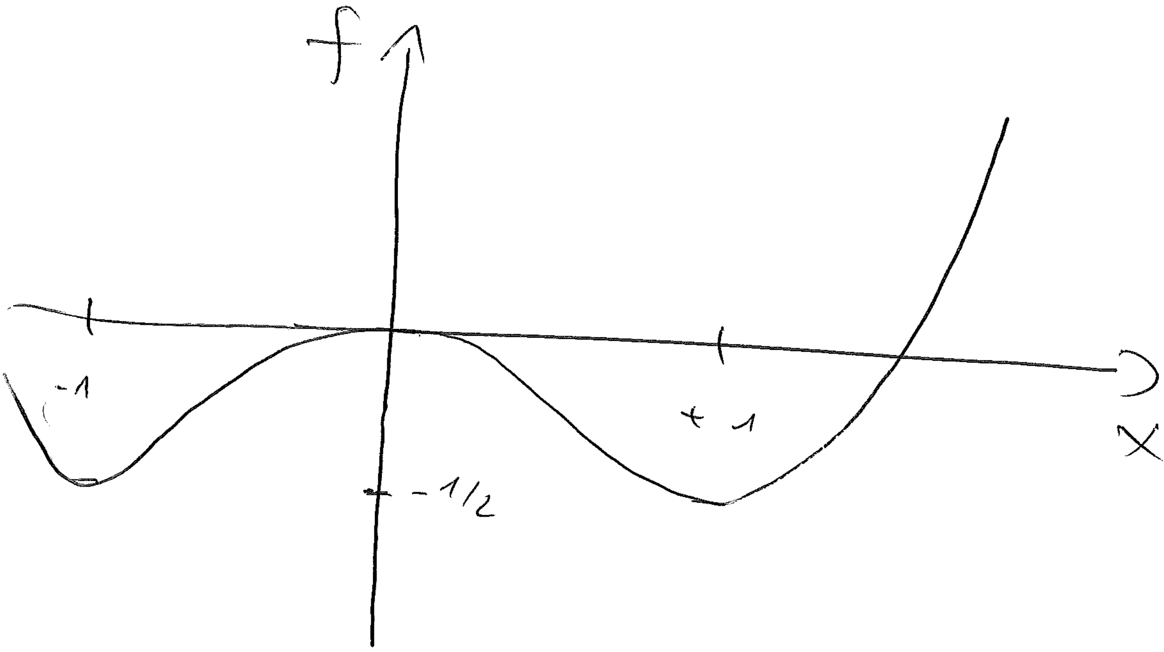
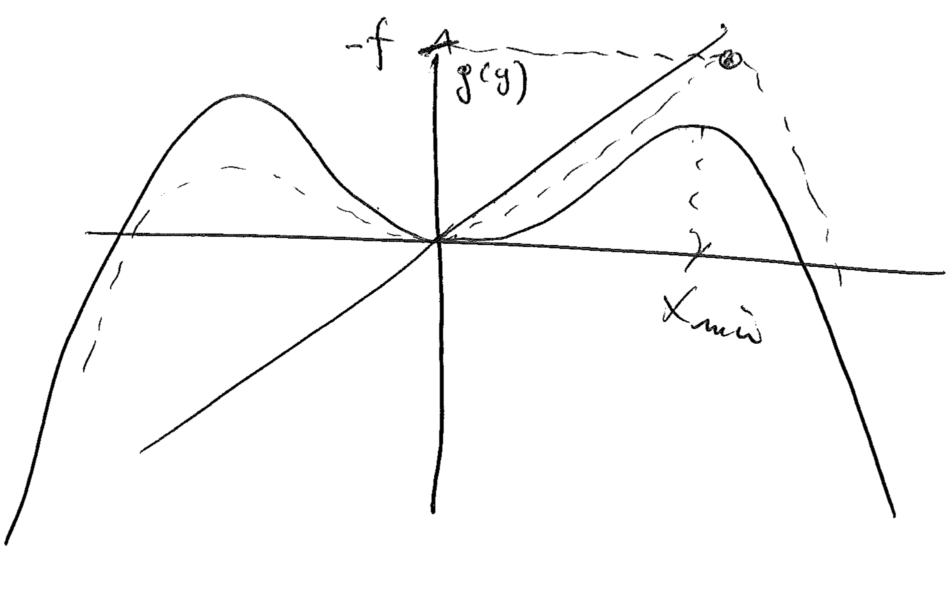
Legendre transformationen

Sei $f(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$



Legendre transformation:

$$g(y) = \max_x \left(y \cdot x - f(x) \right) = \mathcal{L}(f)$$

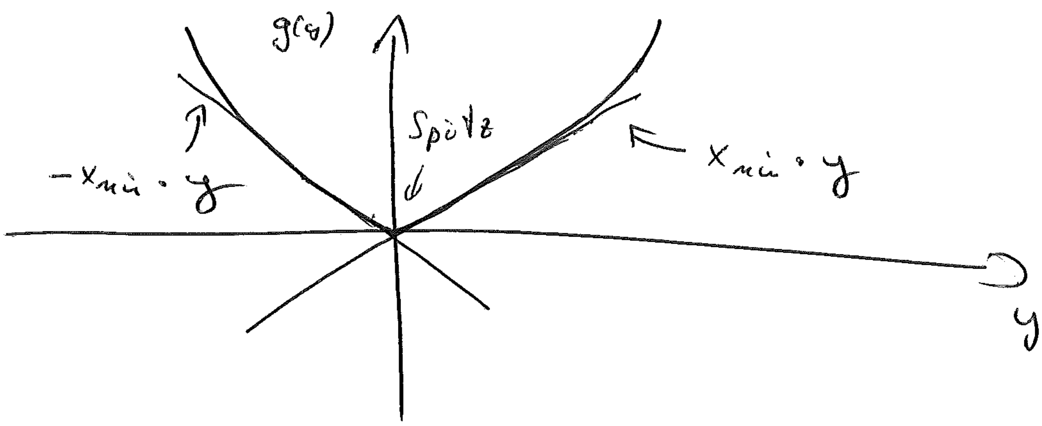


Eigenschaften:

- Maximum: $\frac{\partial}{\partial x}\left(yx - f(x)\right)\Big|_{x_{max}} = 0$

$$\Rightarrow \boxed{y = f'(x_{max}(y))} \qquad (1)$$

- Ableitung: $\boxed{\frac{\partial}{\partial y} g(y) = x_{max}} \qquad (2)$



$y = 0$: $f'(x_{max}) = 0 \quad \leadsto \quad x_{max} = x_{min}$ für $y \to 0_+$

$x_{max} = -x_{min}$ für $y \to 0_-$

$$\Rightarrow \boxed{g'(0_\pm) = \pm x_{min}}$$

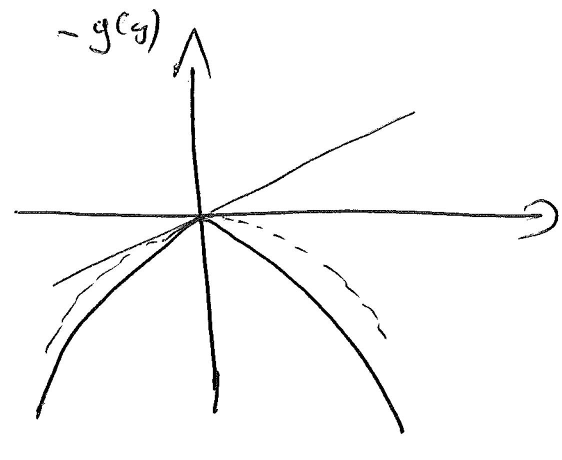
Legendretransformation von g:

$$\bar{f}(x) = \max_y \left(x \cdot y - g(y)\right)$$

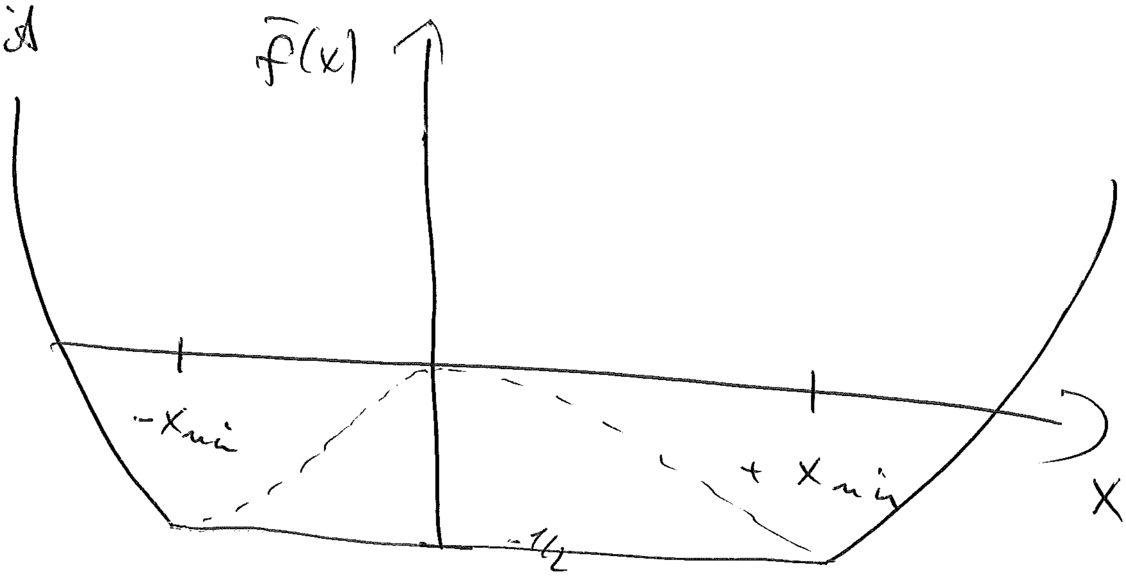
mit

$$\boxed{\begin{aligned} x &= g'(y_{max}) \\ \bar{f}'(x) &= y_{max} = f'(x) \end{aligned}}$$

$\boxed{\text{wenn ableitbar}}$



und damit



- $\bar f$ ist die konvexe Hülle von f

- Legendretransformationen sind <u>konvexe</u> Funktionen

  Für konvexe Funktionen ist die Legendre-transformation eine isomorphe Transformation mit $\mathcal{L}^2 = \mathbb{1}$.

- $\boxed{\text{Ableitbarkeit} \iff \frac{\partial^2 f}{(\partial x)^2} > 0}$

  Dann gilt $\frac{\partial^2 f}{(\partial x)^2} \frac{\partial^2 g}{(\partial y)^2} = 1$

$$= \frac{\partial}{\partial x} \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \frac{\partial g}{\partial y} = \frac{\partial}{\partial x} y \frac{\partial}{\partial y} \frac{\partial g}{\partial y}$$

$$= \frac{\partial}{\partial x} x = 1$$