I-3 Fixed points in the functional RG

Remark: (a) flow vanish identically for $k \to 0$:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \operatorname{Tr} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \dot{R}_k \Big|_{R_k \hat{=} 0} \hat{=} 0$$

(b) Fixed points have to be searched for in dimensionless quantities:

$\boxed{\text{total rescaling of the theory}}$

as $\quad g(\text{scales} \to \lambda \cdot \text{scales}) = \lambda^{d_g} g(\text{scales})$

e.g. scalar theory in 4d: $\hat{g}$ are dim-less

$$\lambda \to \hat{\lambda} = \lambda$$

$$m^2 \to \hat{m}^2 = m^2/k^2$$

$$\phi \to \hat{\phi} = \phi/k$$

$$V_k[\phi] \to \hat{V}_k[\hat{\phi}] = \frac{1}{k^d} \cdot V_k[\hat{\phi} \cdot k]$$

Fixed point: $\partial_t \hat{g}_i = \beta_i(\vec{\hat{g}})$

$$\beta_i(\vec{\hat{g}}_*) = 0$$

e.g. $\vec{\hat{g}} = (\hat{m}, \hat{\lambda})$

Stability: expansion about $\hat{g}_*$:

$$\hat{g}_i = \hat{g}_{*i} + \delta\hat{g}_i$$

$$\Rightarrow \partial_t \hat{g}_i = \underbrace{\beta_i(\vec{\hat{g}}_*)}_{0} + B_{ij}(\vec{\hat{g}}_*)\,\delta\hat{g}_j + \mathcal{O}(\delta\hat{g}^2)$$

with $B_{ij} = \dfrac{\partial \beta_i}{\partial \hat{g}_j}$

Diagonalise $\delta\hat{g}$: $\partial_t \delta\bar{g}_i = b_i \delta\bar{g}_i$ (no sum)

with $B \cdot \vec{e}_i = b_i \vec{e}_i$, $\delta\vec{\hat{g}} = \sum_i \delta\bar{g}_i \vec{e}_i$

$$\Rightarrow \boxed{\delta\bar{g}_i = e^{b_i t}\,\delta\bar{g}_{0i}}$$

Example: ϕ^4 – Theory

Again we resort to dim. less variables.

We rewrite the flow, p. I-24, as

$$\frac{1}{k^d}\,\partial_t V_k = \frac{1}{d}\,\frac{\Omega_d}{(2\pi)^d}\,\frac{1}{1+\partial_\phi^2 V/k^2}$$

and introduce dim. less fields $\hat{\phi}$ & couplings:

$$\hat{V}(\hat{\rho}) = \frac{1}{k^d} V(\rho) \quad \text{with } \rho = \phi^2/2$$

$$\hat{\rho} = \rho \cdot k^{2-d} \quad \text{with } \hat{\phi} = \phi \cdot k^{(2-d)/2}$$

and hence $\left(\partial_{\hat{\phi}}^2 = \partial_{\hat{\rho}} + 2\hat{\rho}\,\partial_{\hat{\rho}}^2,\ \hat{V}' = \partial_{\hat{\rho}}\hat{V}\right)$

$$\frac{1}{k^2}\,\partial_t\big|_\rho \left(\hat{V}\,k^d\right) = \frac{1}{d}\,\frac{\Omega_d}{(2\pi)^d}\,\frac{1}{1+\hat{V}'+2\hat{\rho}\hat{V}''}$$

$$\Rightarrow \boxed{\left(\partial_t\big|_{\hat{\rho}} + d + (2-d)\,\hat{\rho}\,\partial_{\hat{\rho}}\right)\hat{V} = \frac{1}{d}\,\frac{\Omega_d}{(2\pi)^d}\,\frac{1}{1+\hat{V}'+2\hat{\rho}\hat{V}''}}$$

Exercise: How does this generalise to

the $O(N)$ case: ϕ^a, $a = 1,\dots,N$; $V = V[\phi^a\phi^a/2]$

The fixed point equation follows as

$$\boxed{\left(d + (2-d)\,\hat{\rho}\,\partial_{\hat{\rho}}\right)\hat{V} = \frac{1}{d}\frac{\Omega_d}{(2\pi)^d}\frac{1}{1+\hat{V}'+2\hat{\rho}\hat{V}''}}$$

This can only be solved numerically,

for a program (Mathematica) see the webpage of the lecture: (http://www.thphys.uni-heidelberg.de/critical/critical-uebungen.php).

Here we resort to the simplest approximation

$$V = \frac{1}{2}\lambda(\rho - \kappa)^2$$

with

$$\partial_t V(\rho) = \frac{1}{2}\dot{\lambda}(\rho-\kappa)^2 - \lambda\dot{\kappa}(\rho-\kappa)$$

and

$$V' + 2\rho V'' = \lambda(\rho-\kappa) + 2\lambda\rho$$

In these eqs. we have re-named the dim. less quantities with unhatted variables.

The flows for λ, κ are derived from the Taylor expansion of the flow eq. at $V'(\rho_0)=0$

Note that the projection onto the flows of the couplings is <u>not</u> unique, e.g. we could also evaluate the flow at other expansion points with or within a Taylor expansion.

On the lhs this leads to ($\kappa > 0$):

$$\partial_\rho^2\Big|_{\rho=\kappa} \left(\partial_t + d + (2-d)\rho\,\partial_\rho\right)V\Big|_{\rho=\kappa} = \underset{=\dot\lambda}{\beta_\lambda} - (4-d)\,\lambda$$

$$\partial_\rho\Big|_{\rho=\kappa} \left(\partial_t + d + (2-d)\rho\,\partial_\rho\right)V\Big|_{\rho=\kappa} = -\lambda\,\underset{=\dot\kappa}{\beta_\kappa} - (2-d)\,\kappa\lambda$$

rhs:

$$\partial_\rho^2\Big|_{\rho=\kappa} \frac{1}{d}\frac{\Omega_d}{(2\pi)^d}\,\frac{1}{1+V'+2\rho V''}\Big|_{\rho=\kappa} = \frac{18}{d}\frac{\Omega_d}{(2\pi)^d}\,\lambda^2\,\frac{1}{(1+2\lambda\kappa)^3}$$

$$\partial_\rho\Big|_{\rho=\kappa} \frac{1}{d}\frac{\Omega_d}{(2\pi)^d}\,\frac{1}{1+V'+2\rho V''}\Big|_{\rho=\kappa} = -\frac{3}{d}\frac{\Omega_d}{(2\pi)^d}\,\lambda\,\frac{1}{(1+2\lambda\kappa)^2}$$

Finally

$$\boxed{\begin{aligned} \beta_\lambda &= (d-4)\lambda + \frac{18}{d}\frac{\Omega_d}{(2\pi)^d}\lambda^2 \frac{1}{(1+2\lambda k)^3} \\ \beta_k &= (2-d)\,k + \frac{3}{d}\frac{\Omega_d}{(2\pi)^d}\frac{1}{(1+2\lambda k)^2} \end{aligned}}$$

Fixed points: $\boxed{\vec{\beta}_* = \begin{pmatrix} \lambda\beta_k \\ \beta_\lambda \end{pmatrix} = 0}$

(i) Gaußian FP:

$$(k_*, \lambda_*) = (0, 0)$$

Note that $\beta_{k_*} \neq 0$ but the flow for λk was $\lambda\beta_k = \lambda(\dots)$

(ii) Wilson-Fisher FP in 3d:

$$\Omega_d = \frac{2\pi^{3/2}}{\underbrace{\Gamma(3/2)}_{=\sqrt{\pi}/2}} = 4\pi$$

$$\Omega_d/(2\pi)^d = \frac{1}{2\pi^2}$$

The β-fcts. are given by

$$\boxed{\begin{aligned}\beta_\lambda &= -\lambda + \frac{3}{\pi^2}\lambda^2 \frac{1}{(1+2\lambda k)^3}\\ \beta_k &= -k + \frac{1}{2\pi^2}\frac{1}{(1+2\lambda k)^2}\end{aligned}}$$

The stability matrix reads $(\vec{\lambda} = (k, \lambda))$

$$(B_{ij}) = \left(\frac{\partial \beta_i}{\partial g_j}\right) = \begin{pmatrix} -1 - \frac{2}{\pi^2}\frac{\lambda}{(1+2k\lambda)^2} & -\frac{18}{\pi^2}\frac{\lambda^3}{(1+2k\lambda)^4} \\ -\frac{2}{\pi^2}\frac{k}{(1+2k\lambda)^3} & -1+\frac{6}{\pi^2}\frac{\lambda(1-k\lambda)}{(1+2k\lambda)^4}\end{pmatrix}$$

The eigenvalues ω_i $B\cdot\delta\lambda_i = \omega_i\delta\lambda_i$, $i=0,\dots$, follow.

$$\omega_0 = -2, \qquad \omega_1 = 1/3$$

and hence we have

$$\boxed{\nu = 1/2 \;, \quad \omega_1 = -b_2 = 1/3}$$

$$\nu = -1/\omega_0$$

Now we increase the polynomial truncation, and with

$$V_k[\rho] = \sum_{n=1}^{N_{max}=6} \frac{\lambda_n}{n!}(\rho - \kappa)^n$$

we get

$$\nu = 0.647 \quad , \quad \omega = 0.672$$

as compared to the result for $N_{max} \to \infty$.

$$\boxed{\nu = 0.65 \quad , \quad \omega = 0.656}$$

The theoretical value for ν is

$$\nu = 0.63\ldots$$

Remarks:

(i) The value for ν above depends on the regulator. The regulator used, eq. (4.87), p. 133, is optimised and provides the value for ν closest to the exact one.

The range of ν's is given by

$$\boxed{0.65 \leq \nu \leq 0.69}$$

$$\begin{array}{cc} R_{opt} & R_{sharp} \\ \shortparallel & \shortparallel \\ (k^2-p^2)\Theta(k^2-p^2) & p^2\left(\frac{1}{\Theta(p^2-k^2)}-1\right) \end{array}$$

see also script of 'Non-pert. methods in gauge theories' SS 08.

(ii) The final step towards the exact result is the inclusion of the <u>wave function renormalisation</u> $Z_k(p^2, \phi_c)$, see p. I-23.

If the full Z_k could be computed, the flow for V_k would be exact. However, the flow of Z_k requires more than the knowledge of Z_k & V_k.

(iii) The next step leading to a closed set of equation is the approximation to $\Gamma_k^{(2)}$;

$$\boxed{\Gamma_k^{(2)}[\phi_c](p,q) = \left(Z_k\, p^2 + \partial^2_{\phi_c} V_k(\phi_c)\right) \cdot (2\pi)^d\, \delta(p+q)}$$

The flow of Z_k is extracted from

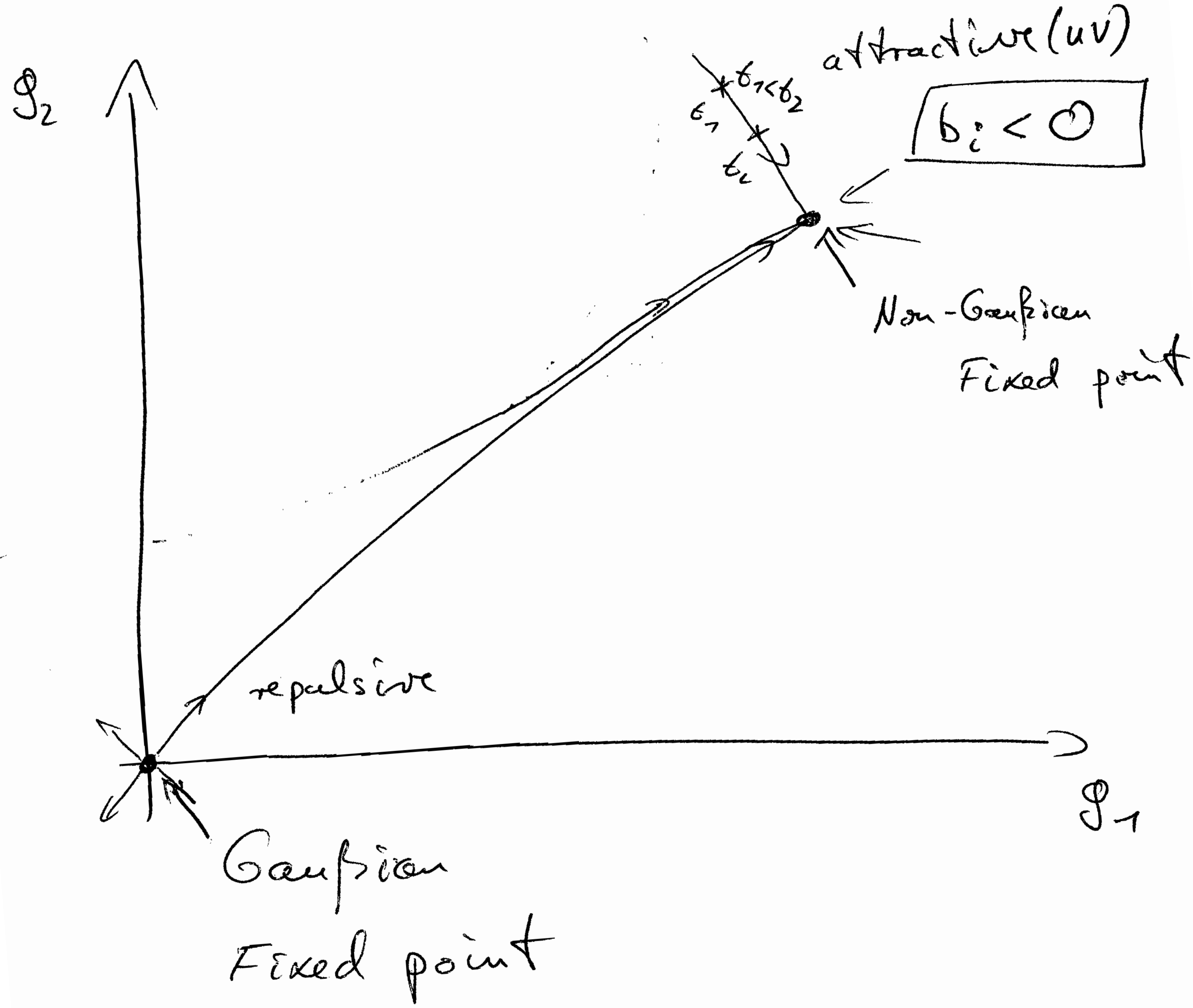
$$\partial_{p^2}\Big|_{p=0} \dot{\Gamma}_k^{(2)}[\phi_c](p,q)$$

and leads to

$$\nu = 0.64\ldots$$

$$\eta = 0.044\ldots$$

in comparison to $\nu = 0.63\ldots$, $\eta = 0.034\ldots$


g_2
attractive (UV)
$t_1 < t_2$
t_1
t_2
$\boxed{b_i < 0}$
Non-Gaußian
Fixed point
repulsive
g_1
Gaußian
Fixed point

b_i complex