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Non-perturbative aspects of gauge theories

Exercise sheet 1

The solutions will be discussed in the tutorials on 13th & 15th of November.

1. Zero-dimensional field theory

Consider the generating function of a bosonic field theory defined on zero spacetime dimensions,

$$Z(j) = \int dx\, e^{-S(x)+jx},$$

where $x \in \mathbb{R}$ are the "fields", j is the source and the classical action $S(x)$ is given by

$$S(x) = \frac{1}{2}x^2 + \frac{\lambda}{4!}x^4.$$

This integral cannot be solved analytically. We want to transform the problem of solving this integral into the problem of solving a (partial) differential equation. To this end we introduce a quadratic "cutoff" term $\frac{1}{2}Rx^2$, $R \geq 0$, to the classical action, which yields a modified generating function:

$$Z(j,R) = \int dx\, e^{-S(x)-\frac{1}{2}Rx^2+jx} = \int dx\, e^{-\frac{1}{2}(1+R)x^2-\frac{\lambda}{4!}x^4+jx}.$$

The question we now ask is: how does the generating function behave under continuous variations of the cutoff-parameter R?

To answer this question it is more convenient to consider the "flowing action" $\Gamma(\bar{x}, R)$, which is defined by a modified Legendre transformation of $\ln Z(j,R)$:

$$\Gamma(\bar{x}, R) = \sup_j \Big(j\bar{x} - \ln Z(j,R) \Big) - \frac{1}{2}R\bar{x}^2 = j_{\sup}\bar{x} - \ln Z(j_{\sup}, R) - \frac{1}{2}R\bar{x}^2,$$

where $\bar{x}$ is the expectation value of x in the presence of the source and the cutoff:

$$\bar{x} = \langle x \rangle_{j,R} = \partial_j \ln Z(j,R).$$

We choose $\bar{x}$ to be independent of R.

(a) First, we want to show that pertubation theory fails at some point for this type of problem.

- Compute the coefficients Z_n of

  $$Z(0) = \int dx\, e^{-\frac{1}{2}x^2-\frac{\lambda}{4!}x^4}$$

  within the perturbative expansion in powers of λ (about $\lambda = 0$),

  $$Z(j) = \sum_n Z_n \lambda^n.$$

  What is the radius of convergence of this expansion?
  Hint: $\int_0^\infty dt\, e^{-t}t^x = \Gamma(x+1)$, *where* Γ *is the Gamma-function.*

– The remainder R_N of the partial sum of order N can be estimated by

$$R_N = \Big| Z(0) - \sum_{n=0}^{N} Z_n \lambda^n \Big| \le \lambda^{N+1} \big| Z_{N+1} \big|$$

(You do not have to prove this). Use the Stirling formula

$$\Gamma(x \to \infty) \to x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi}$$

to estimate $Z_n \lambda^n$ for large n. Estimate the order $N = N_{\rm min}$ in which the above remainder is minimized.

(b) Derive the flow equation for the flowing action:

$$\partial_R \Gamma(\bar{x}, R) = \frac{1}{2} \Big(\partial_{\bar{x}}^2 \Gamma(\bar{x}, R) + R \Big)^{-1}. \qquad (1)$$

Hint: $j_{\rm sup} = j_{\rm sup}(\bar{x}, R)$.

(c) In order to find approximate solutions of eq.(1), we make the following ansatz for the flowing action,

$$\Gamma(\bar{x}, R) = \sum_{n=1}^{N} \frac{\lambda_{2n}(R)}{(2n)!} \bar{x}^{2n},$$

Derive the flow equations for the coefficients $\lambda_2(R)$ and $\lambda_4(R)$, i.e. derive equations for $\partial_R \lambda_2(R)$ and $\partial_R \lambda_4(R)$ from eq.(1). Solve these equations numerically for $N = 2$, that is $\lambda_{2n} \equiv 0$ for $n > N$. Give diagrammatical representations of the equations you found.

2. The generating functional of an interacting QFT

Consider the generating functional (in Euclidean space)

$$Z[j] = \int \mathcal{D}\varphi \, e^{-S_0[\varphi] - S_{\rm int}[\varphi] + \int_x j(x)\varphi(x)},$$

where the free and interatcting part of the classical action are given by:

$$S_0[\varphi] = \int_x \left[\frac{1}{2}(\partial_\mu\varphi)(\partial_\mu\varphi) + \frac{1}{2}m^2\varphi^2\right] \quad \text{and} \quad S_{\rm int}[\varphi] = \int_x \frac{\lambda}{3!}\varphi^3,$$

and we used the abbreviation $\int_x = \int d^4x$. The interacting generating functional can be represented in terms of the free one as

$$Z[j] = e^{-S_{\rm int}[\frac{\delta}{\delta j}]} Z_0[j], \tag{2}$$

where $\frac{\delta}{\delta j}$ is the functional derivative with respect to the source field. The generating functional $Z_0[J]$ is that of the free theory,

$$Z_0[j] = \int \mathcal{D}\varphi \, e^{-S_0[\varphi] + \int_x j(x)\varphi(x)} = Z_0[0] \, e^{-\frac{1}{2}\int_{x,y} j(x)G(x,y)j(y)}\,, \tag{3}$$

with the free field propagator $G(x, y)$,

$$G(x, y) = (-\partial_\mu^2 + m^2)^{-1}\delta^{(4)}(x - y).$$

Prove the eqs.(2) and (3).

Hints:

Represent $f(\varphi)e^{\int_x j(x)\varphi(x)}$ *in terms of derivatives w.r.t.* j.

Substitute $\varphi(x) \to \varphi(x) + \int_z G(x, z)j(z)$.