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Non-perturbative aspects of gauge theories

Exercise sheet 2

The solutions will be presented in the tutorial on 29th of November.

These exercises are based on the chapters 4.3 and 5 of the lecture "critical phenomena" [http://www.thphys.uni-heidelberg.de/~pawlowsk/critical/critical-script.php].

1. Critical exponents of the $O(N)$ model

We want to derive critical exponents of a scalar $O(N)$-symmetric theory. In the local potential approximation the truncation for the flowing action is given by:

$$\Gamma_k[\phi] = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + V_k(\rho) \right], \tag{1}$$

where $\phi = (\phi_1, \ldots, \phi_N)^T$ is an N-component scalar field and $\rho = \phi^2/2$.

(a) Derive the the flow equation of the effective potential $V_k(\rho)$ using the optimized regulator function

$$R_k(q^2) = (k^2 - q^2)\Theta(k^2 - q^2)\mathbb{1}_N, \tag{2}$$

where $\mathbb{1}$ is the $N \times N$ identity matrix in $O(N)$ space.

Note that $V_k(\rho)$ does not contain spacetime-derivatives and therefor we find for spacetime-independent fields ϕ_c:

$$\partial_t \Gamma_k[\phi]\Big|_{\rho=\rho_c} = (\mathrm{Vol}_d)\partial_t V_k(\rho_c). \tag{3}$$

(Vol_d) is the Volume of d-dimensional spacetime. Use the following parametrization for the constant field:

$$\phi_c = \begin{pmatrix} \vec{0} \\ \sigma \end{pmatrix}, \tag{4}$$

where $\vec{0}$ is a $(N-1)$-component zero vector.

Hint:

$$\int \frac{d^d q}{(2\pi)^d} f(q^2) = \frac{\Omega_d}{(2\pi)^d} \int d|q| q^{d-1} f(q^2), \qquad \Omega_d = \frac{2\pi^{d/2}}{\Gamma[d/2]}.$$

(b) Choose the following ansatz for the effective potential:

$$V_k(\rho) = \frac{\lambda_k}{2}(\rho - \rho_{0,k})^2, \tag{5}$$

and derive the flow equations of the couplings λ_k and $\rho_{0,k}$.

(c) Compute the eigenvalues of the stability matrix

$$B_{ab} = \left.\frac{\partial \beta_a}{\partial g_b}\right|_{\vec{g}=\vec{g}^\star} \tag{6}$$

for the case of $d = 3$ spacetime dimensions and $N = 2$ with the flow equations found in **(b)**. $\vec{g}$ is the vector of dimensionless couplings,

$$\vec{g} = \begin{pmatrix} \hat{\rho}_{0,k} \\ \hat{\lambda}_k \end{pmatrix} = \begin{pmatrix} k^{2-d}\rho_{0,k} \\ k^{d-4}\lambda_k \end{pmatrix}, \tag{7}$$

$\vec{\beta}$ is the vector of flow equations of the dimensionless couplings:

$$\vec{\beta} = \begin{pmatrix} \partial_t \hat{\rho}_{0,k} \\ \partial_t \hat{\lambda}_k \end{pmatrix}, \tag{8}$$

and $\vec{g}^\star = (\hat{\rho}^\star_{0,k}, \hat{\lambda}^\star_k)^T$ denotes the (non-trivial) fixed points of the couplings, which are defined via:

$$\left.\partial_t \hat{\rho}_{0,k}\right|_{\hat{\rho}^\star_{0,k}} = 0, \qquad \left.\partial_t \hat{\lambda}_k\right|_{\hat{\lambda}^\star_k} = 0. \tag{9}$$

The eigenvalues of the stability matrix are directly related to critical exponents.

You can download a Mathematica program for the critical exponents of the $O(N)$ model in $d = 3$ dimensions by Michael Scherer at [`http://www.thphys.uni-heidelberg.de/~pawlowsk/critical/3dONmodel.nb`]

2. Large N limit of the $O(N)$ model

We want to study the $O(N)$ model within the approximation used in the previous exercise in the limit $N \to \infty$.

To this end we work with the rescaled effective potential and field:

$$V_k \to \frac{V_k}{(N-1)}, \qquad \rho \to \frac{\rho}{(N-1)} \tag{10}$$

(a) Compute the eigenvalues of the stability matrix for the (dimensionless) flow equations found in **1(b)** for d dimensions in the limit $N \to \infty$ using the rescaled quantities (10).

(b) Solve the flow equation you found in **1(a)** in the large N limit by using the *method of characteristics*[1].

Hint: In order to get a first-order equation, reformulate the flow of the effective potential in terms of $u(\hat{\rho}) = \hat{V}_k'(\hat{\rho})$, where $\hat{V}_k(\hat{\rho}) = k^{-d}V_k(\rho)$ and $\hat{\rho} = k^{2-d}\rho$.

[1] See for example [`http://www.stanford.edu/class/math220a/handouts/firstorder.pdf`]