# Non-perturbative aspects of gauge theories
# Exercise sheet 1 – Dyson-Schwinger Equations


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For this exercise sheet we are mostly considering a scalar theory in the framework of Dyson-Schwinger Equations. The (Euclidean) Lagrangian for this theory is given by

$$\mathcal{L}[\phi] = \frac{1}{2}(p^2 + m^2)\phi^2 + \frac{\lambda}{4!}\phi^4 \,, \tag{1}$$

which is $Z_2$-symmetric. Nevertheless, it can dynamically acquire a finite expectation value $\phi = \langle\phi\rangle$, spontaneously breaking the symmetry. Additionally, we assume the ground-state $\langle\phi\rangle$ to be space-time independent throughout this exercise sheet.

### Exercise 1: Effective potential from Dyson-Schwinger Equations

In the following we are going to calculate a differential equation for the effective potential in two space-time dimensions for a scalar theory. In order to truncate the infinite tower of equations we work in a derivative expansion, i.e. full correlation functions at vanishing momentum are given derivatives of the effective potential. Practically this reduces to

$$\Gamma^{(2)}_{\phi\phi}(p) = p^2 + V^{(2)}(\phi) \tag{2}$$

$$\Gamma^{(3)}_{\phi\phi\phi} = V^{(3)}(\phi) \,, \tag{3}$$

where we have dropped the momentum dependence for all correlation functions $n \geq 3$, since they are momentum independent in leading order.

a) Recollect the derivation of the master equation for this theory.
You should arrive at the following expression (graphical representation see Figure 1.2 in lecture notes)

$$V^{(1)}(\phi) = \frac{\delta S}{\delta\phi} + \frac{\lambda}{2}\phi\int_p G_{\phi\phi}(p) - \frac{\lambda}{3!}\int_{p,q} G_{\phi\phi}(p)G_{\phi\phi}(q)G_{\phi\phi}(p+q)\Gamma^{(3)}_{\phi\phi\phi} \,. \tag{4}$$

b) Calculate the one-loop term in (4).
The integral is not finite and requires regularisation, split the mass term in the classical action into a renormalised mass and counter term

$$\bar{m}^2 = m^2 - \frac{\lambda}{2}\int_p \frac{1}{p^2 + m^2_{\text{ren}}} \,, \tag{5}$$

which renders the integral finite.

c) Calculate the two-loop term in (4).
The integral is finite, but not entirely straightforward to calculate:

- Introduce Feynman parameters

$$\frac{1}{A_1 \dots A_n} = (n-1)! \int_0^1 \mathrm{d}u_1 \dots \int_0^1 \mathrm{d}u_n \frac{\delta\left(1 - \sum_{k=1}^n u_k\right)}{\left(\sum_{k=1}^n u_k A_k\right)^n} \,. \tag{6}$$

- Complete the square first for $q$ and $p$ consecutively , introducing two new momentum variables $k_1, k_2$ that decouple the momentum integrations.
- Rewrite in the denominator as an exponential

$$\frac{1}{D^3} = \frac{1}{2} \int_0^\infty \mathrm{d}t \; t^2 e^{-Dt} \tag{7}$$

and carry out the momentum integrations over $k_1$ and $k_2$.

- Only the integral over the Feynman parameters is left, the result is given by:

$$\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z \frac{\delta(x+y+z-1)}{xy+xz+yz} \tag{8}$$
$$= \frac{1}{18}\left(\psi^{(1)}(1/6) + \psi^{(1)}(1/3) - \psi^{(1)}(2/3) - \psi^{(1)}(5/6)\right) \approx 2.3439 \,,$$

where $\psi^{(n)}(z)$ is the n-th derivative of the digamma function, $\psi^{(0)}(z) = \Gamma'(z)/\Gamma(z)$.

The result you have derived is a differential equation for the effective potential. In order to solve it the initial condition is the classical potential at a large field value, where all quantum fluctuations are suppressed. Solving the differential equation is however in almost all cases analytically not possible.

### Exercise 2: Dyson-Schwinger Equation for the two-point function

*Solving Exercise 3 renders this exercise trivial.*
Derive the Dyson-Schwinger equation of the two-point function for the same theory as in Exercise 1.

**Exercise 3***: **Dyson-Schwinger Equations for arbitrary theories**

Derive the Dyson-Schwinger Equation for the two-point function in the superfield formalism.

The superfield formalism collects all fields in a single superfield, where the field-space metric accounts for possible minus signs for fermions. For example, in Yang-Mills theory we have $\Phi = (A, c, \bar{c})^T$ and the metric reads

$$(\gamma^{ab}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} , \tag{9}$$

which results from the requirement $\Phi^a \Phi_a = A^2 + 2\bar{c}c$, where the delta-distribution in space is implicitly contained in the metric $\gamma$ and integration on the right-hand side is implied. By (our) convention indices are always raised from the left and lowered from the right, e.g. $\Phi^a = \gamma^{ab}\Phi_b = \Phi_b \gamma^{ab} = (A, \bar{c}, -c)$. This immediately implies the following identities

$$\gamma_a{}^b = \gamma^{bc}\gamma_{ac} = \gamma^{cb}\gamma_{ca} = \delta_a^b \tag{10}$$

$$\gamma^a{}_b = \gamma^{ac}\gamma_{cb} = \gamma^{ca}\gamma_{bc} = (-1)^{ab}\delta_b^a \, , \tag{11}$$

where $(-1)^{ab}$ is -1 iff a and b are fermionic and 1 else. For completeness we do not set the fields in the end to zero, but to a finite, arbitrary value. As we cannot go to arbitrary order in the classical action we will restrict ourselves to a polynomial up to order 4, i.e.

$$S[\Phi] = \frac{1}{2!} S^{ij}\Phi_i\Phi_j + \frac{1}{3!} S^{ijk}\Phi_i\Phi_j\Phi_k + \frac{1}{4!} S^{ijkl}\Phi_i\Phi_j\Phi_k\Phi_l \, , \tag{12}$$

where integration is again implied.
With this at hand you can derive the Dyson-Schwinger equation for a two point function starting from

$$\frac{\delta\Gamma}{\delta\Phi_i} = \frac{\delta S}{\delta\varphi_i}\left[\varphi_i = G_{ij}\frac{\delta}{\delta\Phi_j} + \Phi_i\right] . \tag{13}$$