Non-perturbative aspects of gauge theories
Exercise sheet 12 – Quantum Gravity in the Einstein-Hilbert truncation

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Exercise 20: Quantum Gravity in the Einstein-Hilbert truncation

In this exercise we investigate quantum gravity in the Einstein-Hilbert truncation,

$$\Gamma_k = 2\kappa^2 Z_k \int \mathrm{d}^4 x \sqrt{g}\, [2\Lambda_k - R] \,, \tag{1}$$

and $g = \bar{g}$. As a further simplification we only take contributions from the transverse-traceless spin-two mode of the graviton into account, i.e., we neglect the other graviton as well as the ghost modes. Due to this approximation, we never have to specify a gauge-fixing action. Start from transverse-traceless graviton two-point function

$$\Gamma^{(2)}_{h^{\mathrm{TT}} h^{\mathrm{TT}}} = \frac{Z_k}{32\pi} \left(\Delta - 2\Lambda_k + \frac{2}{3} R \right) . \tag{2}$$

We define a completely transverse-traceless regulator

$$R_k = \Gamma^{(2)}_{h^{\mathrm{TT}} h^{\mathrm{TT}}} \Big|_{\Lambda_k = \bar{R} = 0} \cdot r_k \left(\frac{\bar{\Delta}}{k^2} \right) , \tag{3}$$

with the Litim-type cutoff

$$r_k(x) = \left(\frac{1}{x} - 1 \right) \Theta(1 - x) \,. \tag{4}$$

Then the flow equation only includes the TT-part of the propgator. Evaluate now the trace over the Laplace operator on the right-hand side of the Wetterich equation

$$\mathrm{Tr} \left[\frac{1}{\Gamma^{(2)}_k + R_k} \right]_{\mathrm{TT}} \partial_t R_k \,, \tag{5}$$

with heat-kernel techniques, see next page as well as Appendix G.1 of the lecture notes for details.

Turn now to the left-hand side of the Wetterich equation and take a scale derivative of (1). Compare the terms proportional to $\sqrt{g}$ and $\sqrt{g}R$ from the left-hand side with the result from the right-hand side, (5). Deduce from this the flow equations of the Newton coupling and the cosmological constant. The resulting flow equations are

$$\begin{aligned}\partial_t g_k &= (2+\eta_g) g_k \,,\\ \eta_g &= -\frac{5}{6\pi} g_k \left(2\frac{1-\frac{1}{6}\eta_g}{(1-2\lambda_k)^2} + \frac{1-\frac{1}{4}\eta_g}{1-2\lambda_k} \right) ,\\ \partial_t \lambda_k &= -4\lambda_k + \frac{\lambda_k}{g_k}\partial_t g_k + \frac{5}{4\pi} g_k \frac{1-\frac{1}{6}\eta_g}{1-2\lambda_k} \,. \end{aligned} \tag{6}$$

Bonus question 1:
Why does the spin-two approximation work rather well in most cases?

Bonus question 2:
Use mathematica and find numerically the non-Gaussian fixed point of (6) and determine the eigenvalues of the stability matrix.

Heat-kernel techniques

Heat-kernel techniques are used to evaluate the trace of a function that depends on the Laplace operator on a curved background. You can use the formula

$$\mathrm{Tr}\, f(\Delta) = \frac{1}{(4\pi)^2}\Big[B_0(\Delta) Q_2[f(\Delta)] + B_2(\Delta) Q_1[f(\Delta)] \Big] + \mathcal{O}(R^2)\,, \tag{7}$$

with the definition

$$Q_n[f(x)] = \frac{1}{\Gamma(n)} \int \mathrm{d}x\, x^{n-1}\, f(x)\,. \tag{8}$$

The B_n are called heat-kernel coefficients and often written as

$$B_n(\Delta) = \int \mathrm{d}^4x \sqrt{g}\;\; \mathrm{Tr}\, b_n(\Delta)\,. \tag{9}$$

The values of the heat-kernel coefficients depend on the field. For the transverse-traceless tensor (TT), transverse vectors (TV) and scalars (S) on a sphere, they are given by

	TT	TV	S
Tr b_0	5	3	1
Tr b_2	$-\frac{5}{6}R$	$\frac{1}{4}R$	$\frac{1}{6}R$