Non-perturbative aspects of gauge theories
Exercise sheet 4 – $O(N)$ model at large N

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Exercise 9: Analytic solution of the $O(N)$ model at large N

The flow equation of the $O(N)$ model with a flat cutoff was derived in *Exercise 4* and is given by

$$\partial_t V(\rho) = A_d k^{d+2} \left(\frac{N-1}{k^2 + V_k'(\rho)} + \frac{1}{k^2 + V_k'(\rho) + 2\rho V_k''(\rho)} \right) , \tag{1}$$

with $A_d = d^{-1}(2\pi)^{-d}\Omega_d$. This equation can be solved analytically at $N \gg 1$. In order to do so rescale the potential $V(\rho)$ and the invariant ρ with $N-1$. Drop all terms $\sim N^{-1}$ and convince yourself that the resulting equation can be written as

$$\partial_t V(\rho) = A_d k^{d+2} \left(\frac{N-1}{k^2 + V_k'(\rho)} \right) . \tag{2}$$

Derive, starting from (2), an equation for $w(\rho) = V_k'(\rho)$ and bring it into quasilinear form, i.e.

$$a(t,\rho,w)\partial_t w + b(t,\rho,w)\partial_\rho w = c(t,\rho,w) . \tag{3}$$

Equations of the form (3) are suitable for the *method of characteristics*, a small practical introduction to this method can be found in the lecture notes (or any other suitable source of your liking). The resulting system of ordinary differential equations you should find is

$$\begin{aligned}
&\partial_s t = 1 , \\
&\partial_s \rho = A_d \left[\Lambda e^s\right]^{d+2} \frac{1}{\left(\left[\Lambda e^s\right]^2 + w\right)^2} , \\
&\partial_s w = 0 .
\end{aligned} \tag{4}$$

Formulate suitable initial conditions and solve (4), while a fully analytic representation is possible, it is not insightful here, therefore you can keep the solution for ρ as an indefinite integral. With this solution one obtains an implicit relation $\rho = \rho(t, \rho_0)$ and the derivative of the potential is given by $w(t,\rho) = w_0(t, \rho_0(\rho))$. In summary, to get the derivative of the potential at some RG time t, one needs to invert the implicit relation $\rho = \rho(t, \rho_0)$, which analytically not possible in this case. Nevertheless, physical information can be extracted, as a final calculation derive the RG time dependent minimum $\rho_{\mathrm{min}}(t)$. In order to do so utilize the condition for the minimum $w = 0$ to solve the remaining integral analytically.