Non-perturbative aspects of gauge theories
Exercise sheet 6 – Quark-Meson Model

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Exercise 12: Phase structure of the Quark Meson Model

In this exercise you are going to calculate parts of the phase structure of the Quark Meson Model introduced in the lecture.

In order to do this, recollect the derivation of the flow equation at finite temperature and density for the two flavour Quark Meson Model using a flat regulator in Section 3.2.4 of the script. For this exercise we are staying in the chiral limit, i.e. setting the bare quark mass $m_q = 0$. You should arrive at the result

$$\partial_t V(\rho) = \frac{k^5}{12\pi^2}\Bigg\{ \sum_{\varphi\in\{\pi,\sigma\}} \frac{1}{\epsilon_k^\varphi}\left[1 + 2n_{\mathrm{B}}\left(\epsilon_k^\varphi\right)\right] - \frac{N_f N_c}{\epsilon_k^q}\left[1 - n_{\mathrm{F}}\left(\epsilon_k^q + \mu_q\right) - n_{\mathrm{F}}\left(\epsilon_k^q - \mu_q\right)\right]\Bigg\}, \tag{1}$$

where $N_c = 3$ and $N_f = 2$. The dispersion relations are given by

$$\left(\epsilon_k^\pi\right)^2 = k^2 + V'(\rho) \tag{2}$$

$$\left(\epsilon_k^\sigma\right)^2 = k^2 + V'(\rho) + 2\rho\, V''(\rho) \tag{3}$$

$$\left(\epsilon_k^q\right)^2 = k^2 + h_\sigma \left(\frac{\rho}{2}\right)^{\frac{1}{2}} . \tag{4}$$

The occupation numbers in (1) are given by

$$n_{\mathrm{B}}(z) = \left(e^{z/T} - 1\right)^{-1} \tag{5}$$

$$n_{\mathrm{F}}(z) = \left(e^{z/T} + 1\right)^{-1} . \tag{6}$$

The remaining part of the exercise requires the usage of a Computer-Algebra system with numerical capabilities, e.g. Mathematica.

For the actual computation we will only consider finite temperature and set the chemical potential to zero for simplicity, i.e. $\mu_q = 0$. In order to solve (1) we will expand the potential $V(\rho)$ around some fixed expansion point in a Taylor series

$$V(\rho) = \sum_{j=0}^{\mathrm{N_{max}}} \frac{\lambda_j}{j!}(\rho - \kappa)^j \,, \tag{7}$$

please note that you can neglect the $j = 0$ component as it will not couple back into the equation. The full k-dependence is therefore carried by the $\mathrm{N_{max}}$ Taylor coefficients λ_j. Plugging (7) into (1), taking $0 \ldots \mathrm{N_{max}}$ derivatives and evaluating the equation at κ results in a coupled system of ordinary differential equations. Practically, we you should chose $\mathrm{N_{max}}$ between 5 and 7 for good results. Do not attempt to do this by hand! since the resulting expression get rather large.

We are left with the specification of the initial conditions at the cut off scale $\Lambda = 700\,\mathrm{MeV}$. We start with a quadratic potential in ρ, i.e. $\lambda_j = 0$ for $j \geq 3$. The remaining parameters are adjusted in the vacuum, i.e. $h_\sigma, \lambda_1, \lambda_2, f_\pi$. Adjust your parameters such that you get a minimum at $f_\pi = 88\,\mathrm{MeV}$, a constituent quark mass of $m_q = 300\,\mathrm{MeV}$ and a sigma meson mass of $m_\sigma = 300..500\,\mathrm{MeV}$ and the pion mass vanishes by definition. We don't specify the sigma mass precisely, since it involves fine tuning the initial conditions. The expansion point κ should always be chosen slightly above the IR minimum of the potential, i.e. $\kappa = (88 + \varepsilon)\,\mathrm{MeV}$ in the vacuum and needs to be dynamically adjusted at finite temperature.

With this at hand you can calculate the minimum of the potential as well as the masses as a function of temperature. The second order phase transition should be clearly visible in a plot. You have just calculated the chiral phase transition of QCD in an Low Energy Effective Theory!

Additional hints, remarks and bonus questions:

- Always evaluate all field dependent quantities, e.g. masses, at the expansion point, especially when tuning λ_1 and λ_2 in order to adjust the position of the minimum and the sigma meson mass.
- Calculate for different temperatures iteratively and readjust the expansion point κ starting from your previous temperature.
- Choosing your expansion point κ below the minimum results in an unstable numerical scheme, leading to divergences in your numerical result.

- The flow equation (1) is numerically unstable for $T \to 0$, but the limit can easily be achieved analytically.
- Once you have the potential as a function of temperature it is easy to extract thermodynamic quantities, what is happening at large temperatures?