Non-perturbative aspects of gauge theories
Exercise sheet 8 – Yang-Mills propagators II

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Exercise 14: Derivation of the flow equations for ghost and gluon

Recapitulate Exercise 13 and make sure you understand each step in the derivation on a conceptual level. All details can be found in the script in Section 5.5.3 and Appendix F.3.

We introduce an angular variable $p\,q = p_v q_v z$, which ranges from $-1 \leq z \leq 1$ and the subscript v denotes the modulus of a vector. The regulator derivative is abbreviated as $dR_\varphi = \partial_t r_\varphi - \eta_\varphi r_\varphi$. The scalar parts of the propagators are

$$\bar{G}_{AA}(p) = Z_A(p)^{-1}\left(p^2[1 + r_A(p^2/k^2)] + m_A^2\right)^{-1} \tag{1}$$

$$\bar{G}_{\bar{c}c}(p) = Z_c(p)^{-1}\left(p^2[1 + r_c(p^2/k^2)]\right)^{-1}\,. \tag{2}$$

Please note that the ordering of the indices for the ghost-gluon vertex is different from the one used in the lecture.

The resulting flow equation for the ghost dressing function is

$$\partial_t Z_c(p) = -3(z^2-1)\Bigg\{ dR_A(q)\bar{G}_{AA}(q)^2\bar{G}_{\bar{c}c}(p+q)\lambda_{A\bar{c}c}(-q,p+q)\lambda_{A\bar{c}c}(q,p) \tag{3}$$
$$+ \frac{q_v^2}{p_v^2+q_v^2+2p_vq_vz} dR_c(q)\bar{G}_{AA}(p+q)\bar{G}_{\bar{c}c}(q)^2\lambda_{A\bar{c}c}(-p-q,p)\lambda_{A\bar{c}c}(p+q,-q)\Bigg\}\,.$$

And for the gluon

$$\partial_t\left[Z_A(p)(p^2+m_A^2)\right] = \left(z^2-7\right) dR_A(q)\bar{G}_{AA}(q)^2\gamma_{AAAA}(p,-p,q) \tag{4}$$
$$+ 2q_v^2\left(z^2-1\right) dR_c(q)\bar{G}_{\bar{c}c}(q)^2\bar{G}_{\bar{c}c}(p+q)\lambda_{A\bar{c}c}(-p,-q)\lambda_{A\bar{c}c}(p,-p-q)$$
$$- \frac{4(z^2-1)(3p_v^4+3q_v^4+6p_v^3q_vz+6p_vq_v^3z+p_v^2q_v^2(8+z^2))}{p_v^2+q_v^2+2p_vq_vz}\times$$
$$\times\; dR_A(q)\bar{G}_{AA}(q)^2\bar{G}_{AA}(p+q)\lambda_{AAA}(-p,p+q)\lambda_{AAA}(p,q)\,.$$

Proceed by dressing all vertices appropriately and introduce a single coupling $\alpha_s(\bar{p})$ according to (5.54) in the lecture notes. Use the full momentum dependence in the dressings, such that they cancel in all equations after introducing η on the LHS of the equations and set the argument of the coupling to the symmetric point configuration afterwards. It is give by

$$\bar{p} = \left(\frac{2}{3}\left(p_1^2 + p_2^2 + p_1\, p_2\right)\right)^{\frac{1}{2}} \tag{5}$$

$$\bar{p} = \left(\frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2 + p_1\, p_2 + p_1\, p_3 + p_2\, p_3\right)\right)^{\frac{1}{2}} ,$$

for three- and four-point functions, respectively.

Exercise 15: Solving the propagator flow equations in Yang-Mills

Solve the equations in Exercise 14! For this we have to chose m_A^2, where we will make the choice

$$m_A^2 = \lim_{p\to 0} \mathcal{P}^{(1)}_{AA}\Gamma^{(2)}_{AA} , \tag{6}$$

ensuring a positive Z_A and resulting in the constraint $Z_A(p=0) = 1$. You can get the strong coupling α_s from the wave functions as

$$\bar{\alpha}_s(p) = \frac{1}{4\pi^2}\frac{g^2}{Z_A(p)Z_c(p)^2} , \tag{7}$$

more details are in the lecture notes, c.f. the discussion around (5.75). As initial conditions we now require the UV scale Λ as well as the coupling g and the initial gluon mass parameter m_A^2. The first two are in a sense arbitrary, as we still need to set the scale in the calculation. Practically, use a small, but not too small value for g, e.g. $g = 0.1$, at $\Lambda = 50$, where the units will turn out to be roughly GeV. After calculations determine maximum in the propagator dressing $1/Z_A(p)$ (if there is one) and rescale all units such that the position of the maximum is at 1 GeV, which defines the scale. Solve the flow equations for various choices of $m^2_{A,k=\Lambda} < 0$. Explore the behaviour of $m^2_{A,k=0}(m^2_{A,k=\Lambda})$, where you should find a critical value, corresponding to a Landau-pole-like singularity. You should find the qualitative behaviour shown in Fig. 9 in Cyrol, Fister, Mitter, Pawlowski, Strodthoff (2016).

Practically, you have to solve the differential equations

$$\partial_t m_A^2 = \lim_{p\to 0} \mathcal{P}^{(1)}_{AA}\, \partial_t\Gamma^{(2)}_{AA} \tag{8}$$

$$\partial_t Z_\varphi = -Z_\varphi \eta_\varphi$$

and before each evaluation of the right-hand side of the different equation, solve the coupled integral equations for the anomalous dimensions η_φ. As shape function for the regulator you can use for both fields an exponential regulator

$$r_\varphi(x) = \frac{x\, e^{-x^2}}{1 - e^{-x^2}} , \tag{9}$$

allowing you to limit the radial integration to $q_v \lesssim 3k$.

Exercise 16*: Solving the propagator DSEs in Yang-Mills

Now that you've warmed up, repeat Exercise 13 and Exercise 15 for the corresponding Dyson-Schwinger Equations.