Hydrodynamics

Stefan Flörchinger (Heidelberg)

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What is Hydrodynamics?

Describes the evolution of physical systems (classical or quantum particles, fluids or fields) close to thermal equilibrium.

Foundations of statistical physics

- Ergodic hypothesis: Time averages are equal to ensemble averages

$$\bar{A} = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, A = \mathrm{Tr}\, \rho A = \langle A \rangle.$$

- The microcanonical ensemble depends on the conserved quantities of the system, e.g.

$$\rho = \frac{1}{\Omega(E)} \delta(H - E).$$

- Subsystems that can exchange energy or energy and particles with some larger system are described by the canonical or grand-canonical ensemble

$$\rho = \frac{1}{Z} e^{-\beta H}, \qquad \rho = \frac{1}{Z} e^{-\beta(H-\mu N)}.$$

Equilibration

- Time scale for equilibration is the relaxation time τ.
- τ is typically short when interactions between particles or modes of a field are large.
- For large systems one may have different relaxation times: Subsystems approach equilibrium quite fast, while equilibration for the whole systems needs more time.
- Local equilibrium is described by a distribution with $T = T(\vec{x}, t), \mu = \mu(\vec{x}, t)$, etc.
- Collective motion is described by fluid velocity $\vec{v} = \vec{v}(\vec{x}, t)$.

Hydrodynamical variables

- velocity $\vec{v} = \vec{v}(\vec{x}, t)$,
- mass density $\rho = \rho(\vec{x}, t)$,
- pressure $P = P(\vec{x}, t)$.

For relativistic system the velocity gets replaced by the four-velocity u^μ and the mass density by the inner energy ϵ.
In principle one can replace the thermodynamic variables ρ and P by some other independent pair such as μ and T.

Conservation laws

Conservation of energy and momentum

$$\partial_\nu T^{\mu\nu} = 0,$$

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & \vec{j}_\epsilon \\ \vec{g} & \Pi \end{pmatrix} = \begin{pmatrix} \text{energy density} & \text{energy flux density} \\ \text{momentum density} & \text{momentum flux density} \end{pmatrix}$$

or in components

$$\frac{\partial}{\partial t}\epsilon + \vec{\nabla}\vec{j}_\epsilon = 0 \quad \text{(energy conservation)},$$

$$\frac{\partial}{\partial t} g_i + \nabla_j \Pi_{ij} = 0 \quad \text{(momentum conservation)}.$$

Conservation of mass density current $j^\mu = (\rho, \vec{j})$

$$\partial_\mu j^\mu = 0 \quad \text{or} \quad \frac{\partial}{\partial t}\rho + \vec{\nabla}\vec{j} = 0.$$

Constituive equations

To use the conservation laws one has to express the conserved currents in terms of the hydrodynamical variables,

$$T^{\mu\nu} = T^{\mu\nu}(\vec{v}, \rho, P, \vec{\nabla} \cdot \vec{v}, \vec{\nabla} \times \vec{v}, \vec{\nabla}\rho, \dots),$$

$$J^{\mu} = J^{\mu}(\vec{v}, \rho, P, \vec{\nabla} \cdot \vec{v}, \vec{\nabla} \times \vec{v}, \vec{\nabla}\rho, \dots).$$

- The form of this is constraint by space-time symmetries (Rotation and Galilean or Lorentz boosts).
- The basis idea of hydrodynamics is to expand the above equations in terms of derivatives of the thermodynamic variables. The lowest order of this expansion gives ideal hydrodynamics, the higher orders dissipative corrections.

Non-relativistic ideal fluid 1

- Lowest order is fixed by Rotation invariance, Galilean invariance and conservation of entropy

$$\epsilon = \epsilon_0 + \frac{1}{2}\rho\vec{v}^2, \qquad \vec{j}_\epsilon = \vec{v}(\epsilon + P),$$
$$\vec{g} = \rho\vec{v}, \qquad \Pi_{ij} = P\delta_{ij} + \rho v_i v_j,$$
$$J^\mu = (\rho, \rho\vec{v}).$$

  ϵ_0 is the energy density in the fluid rest frame.

- Conservation laws for energy, momentum and mass density give five equations for the six variables

$$\vec{v}, \rho, P, \epsilon_0.$$

- To close this one needs as an input from thermodynamics the equation of state, e.g. in the form

$$\epsilon_0 = \epsilon_0(\rho, P).$$

Non-relativistic ideal fluid 2

The expressions for $T^{\mu\nu}$ and J^μ plugged into the conservation laws give the hydrodynamic equations. For example, the conservation of momentum density leads to

$$\frac{\partial}{\partial t}\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}P.$$

This is the Euler equation.

Non-relativistic viscous fluid

At the next level of the hydrodynamic derivative expansion some terms of the energy-momentum tensor get modified.

- momentum flux density or stress tensor

$$\Pi_{ij} = P\delta_{ij} + \rho v_i v_j + \delta\Pi_{ij}$$

$$\delta\Pi_{ij} = -\eta(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{v}) - \zeta\delta_{ij}(\vec{\nabla}\vec{v}).$$

- energy flux density

$$(j_\epsilon)_i = v_i(\epsilon + P) + (\delta\Pi_{ij})v_j - \kappa\nabla_i T.$$

Here, η is the shear viscosity, ζ is the bulk viscosity, κ is the thermal conductivity.

Transport coefficients

- Similar to the equation of state, the functions

$$\eta = \eta(\rho, P), \quad \zeta = \zeta(\rho, P), \quad \kappa = \kappa(\rho, P),$$

have to be determined from the underlying microscopic theory. Alternatively, one can fix them from experiments.

- The second law of thermodynamics implies

$$\eta, \zeta, \kappa \geq 0.$$

Navier-Stokes equation

Including the viscous correction to Π_{ij} in the conservation law for the momentum density, one arrives at the Navier-Stokes equation

$$\rho\left[\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}\right]=-\vec{\nabla}P+\eta\Delta\vec{v}+\left(\zeta+\frac{\eta}{3}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{v}).$$

- For incompressible fluids where $\vec{\nabla}\cdot\vec{v}=0$ the last term vanishes and the bulk viscosity ζ drops out.
- The Navier-Stokes equation is not easy to solve since it is a non-linear equation for the velocity field $\vec{v}$.
- Example: Oscillating wall in an incompressible fluid.

Shear viscosity

- The shear viscosity η can have very different values for different fluids.
- The ratio η/s with entropy density s has units $\hbar/k_B$ and might be constraint by a universal lower bound.
- η/s becomes small when interaction / fluctuation effects are large.

Fluid		P (Pa)	T (K)	η (Pa s)	η/n ($\hbar$)	η/s ($\hbar/k_B$)
H_2O	(boiling point)	0.1×10^6	370	2.9×10^{-4}	85	8.2
^4He	(lambda transition)	0.1×10^6	2.0	1.2×10^{-6}	0.5	1.9
H_2O	(tricritical point)	22.6×10^6	650	6.0×10^{-5}	32	2.0
^4He	(tricritical point)	0.22×10^6	5.1	1.7×10^{-6}	1.7	0.7
^6Li ($a = \infty$)		12×10^{-9}	23×10^{-6}	$\leqslant 1.7 \times 10^{-15}$	$\leqslant 1$	$\leqslant 0.5$
QGP		88×10^{33}	2×10^{12}	$\leqslant 5 \times 10^{11}$		$\leqslant 0.4$

Superfluid hydrodynamics 1

Hydrodynamic equations for superfluids were derived by Lev D. Landau.

- Divide fluid into normal and superfluid part

$$\rho = \rho_n + \rho_s$$

  with velocities $\vec{v}_n$ and $\vec{v}_s$.

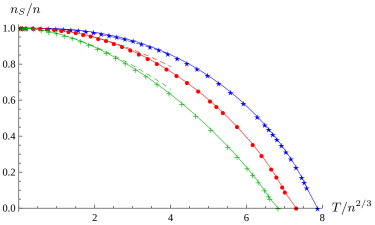
- Superfluid velocity is gradient of the phase of the macroscopic wave function

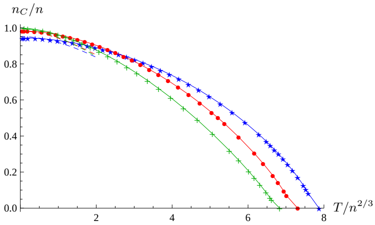
$$\vec{v}_s = \frac{1}{m}\vec{\nabla}\varphi, \quad \vec{\nabla} \times \vec{v}_s = 0.$$

- Entropy is carried only by the normal part of the fluid. For ideal hydrodynamics it is conserved.

Normal and superfluid density

- Superfluid density and condensate density are not the same.
- At zero temperature $\rho = \rho_s$, $\rho_n = 0$.
- At the phase transition $\rho = \rho_n$, $\rho_s = 0$.





Theoretical results for Bose gas with different interaction strength
(S. Floerchinger and C. Wetterich, PRA 79, 063602 (2009)).

Superfluid hydrodynamics 2

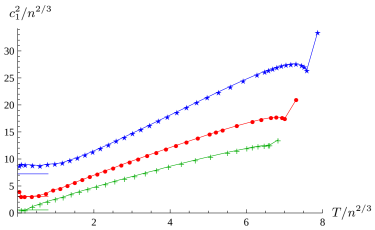
- Hydrodynamic equations are again derived from conservation of energy, momentum und particle number. As an additional equation one has

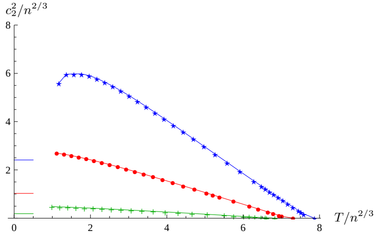
$$\frac{\partial \vec{v}_s}{\partial t} + \vec{\nabla}\left(\frac{\vec{v}_s^2}{2} + \mu\right) = 0.$$

- Besides the equation of state one needs also the ratio ρ_s/ρ as function of thermodynamic variables from the microscopic theory.
- Dissipative superfluid hydrodynamics includes additional viscosity coefficients ζ_2, ζ_3 besides the usual shear viscosity η and bulk viscosity ζ.

Two velocities of sound

- An interesting feature of superfluid hydrodynamics is that sound propagation has two modes with different velocities.
- One mode corresponds to the usual pressure / density wave.
- The other is an oscillation of the superfluid and normal density against each other.





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Relativistic hydrodynamics 1

- For ideal relativistic fluid the energy-momentum tensor is ($g = \text{diag}(1, -1, -1, -1)$)

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu}$$

- In the fluid rest-frame one has $u^\mu = (1, 0, 0, 0)$ and therefore

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}.$$

- Conservation laws for currents read $\partial_\mu J^\mu = 0$ but sometimes the currents simply vanish (as in QCD at $\mu = 0$).

Relativistic hydrodynamics 2

- The derivation of ideal hydrodynamic equations is similar to the non-relativistic case.
- Dissipative relativistic hydrodynamics has some subtleties:
  - Fluid velocity u^μ is not completely unique. Landau frame corresponds to $T^{0i} = 0$ in the rest frame, Eckart frame to $u^\mu \sim J^\mu$.
  - First order dissipative hydrodynamics has causality problems. Numerical solutions get unstable. These problems can be cured by going to second order.
- Numerical simulations of relativistic ideal and dissipative hydrodynamics are used in astrophysics or to describe heavy ion experiments.