Introduction
to
Phase Transitions

D. Dubbers

1. Introduction

Phase transitions in Heidelberg physics dep't.:

- Particle physics — Standard model …
- Nuclear physics — Quark-gluon transition
  Nuclear liquid-gas transition …
- Atomic physics — Bose-Einstein
  Laser
- Condensed matter — Glass transition
  Surfaces
  Biophysics …
- Environmental physics — Condensation
  Aggregation
  Percolation …
- Astrophysics, Cosmology → next page

History of the universe

= Succession of phase transitions

of the vacuum:

Transition	Temperature	Time
Planck	10^{19}eV	~0s
GUT's	?	?
Inflation	?	?
Electro-weak	100GeV	10^{-12}s

of matter, i.e. freeze out of:

Quark-gluon plasma to nucleons	100GeV?	10^{-12}s ?
Nucleons to nuclei	1MeV	1s
Atoms	10eV	10^{5}a
Galaxies	3K	today

Topics not treated

other phase transitions:

Bose-Einstein condensates
Superfluidity
Quantum phase transitions
Aggregates
Fragmentation
Percolation
Liquid crystals
Isolator-metal transitions
Topological defects
Traffic jams
…

other 'critical phenomena' (non-linear physics)

Route to chaos
Turbulence
Self organized criticality (forest fires, avalanches, …)
…

2. Phenomenology: control parameter

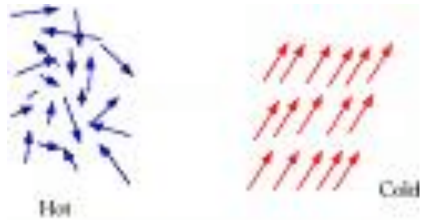
Phase transition = sudden change of the state of a system
upon variation of an external control-parameter:
which reaches 'critical value'.

Here: control-parameter is temperature
(it can also be pressure, atomic composition, connectivity, traffic density, public mood, taxation rate, …):

example: magnet

T_C = critical temperature:

above T_C: paramagnet **PM**
below T_C: ferromagnet **FM**



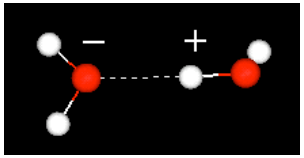
here: T_C = Curie temperature

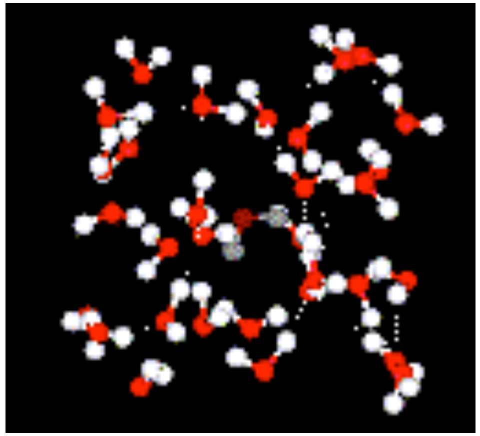
critical phenomena

Why are phase transistions so sudden?

example: liquid

below T_C: liquid **L**
above T_C: gas **G**





Example of boiling water:

bond between 2 molecules breaks due to thermal fluctuation
increased probability that 2nd bond breaks, too: chain reaction

below T_C: broken bond heals, before 2nd bond breaks – water in boiler is noisy
above T_C: broken bond does not heal, before 2nd bond breaks – water boils:

A 'run-away' or 'critical' phenomenon: **L** → **G**

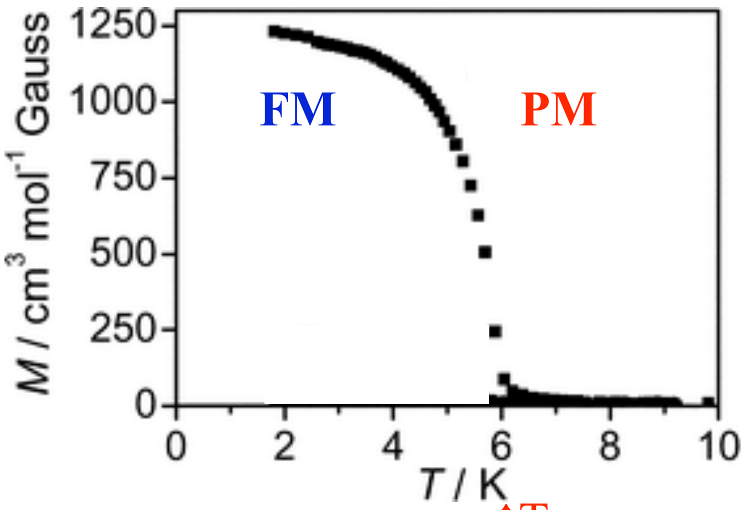
order parameter

At T_C the probe acquires a new property,
the order-parameter M:

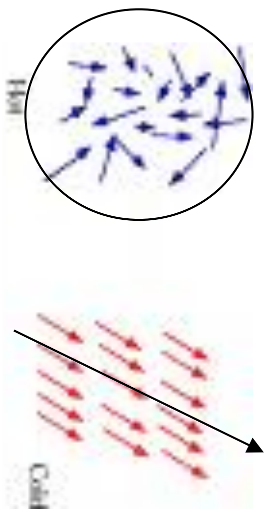
above T_C: $M=0$.
below T_C: $M \neq 0$,

here = magnetisation ***M***

N.B.: Disorder: high symmetry: S
↓ ↓ ∪
Order: low symmetry: S'



↑T_C



PM: rotational symmetry
∪
FM: cylindrical symmetry

critical exponent

Observation:

At $T \lessapprox T_C$ the order M parameter depends on temperature T like:

below T_C: $M(T) = M_0\,(1-T/T_C)^{\beta}$

with critical exponent β

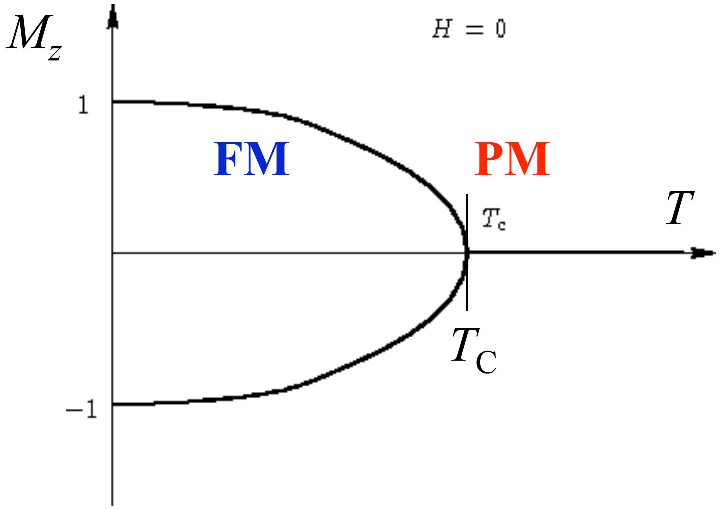
Examples: $M(T) \sim \sqrt{T_C - T}$:

critical exponent $\beta = ½$

$M(T) \sim 3^{rd}\sqrt{T_C - T}$:

critical exponent $\beta = ⅓$

1-dimensional magnet:
"bifurcation"



reduced temperature

With reduced temperature

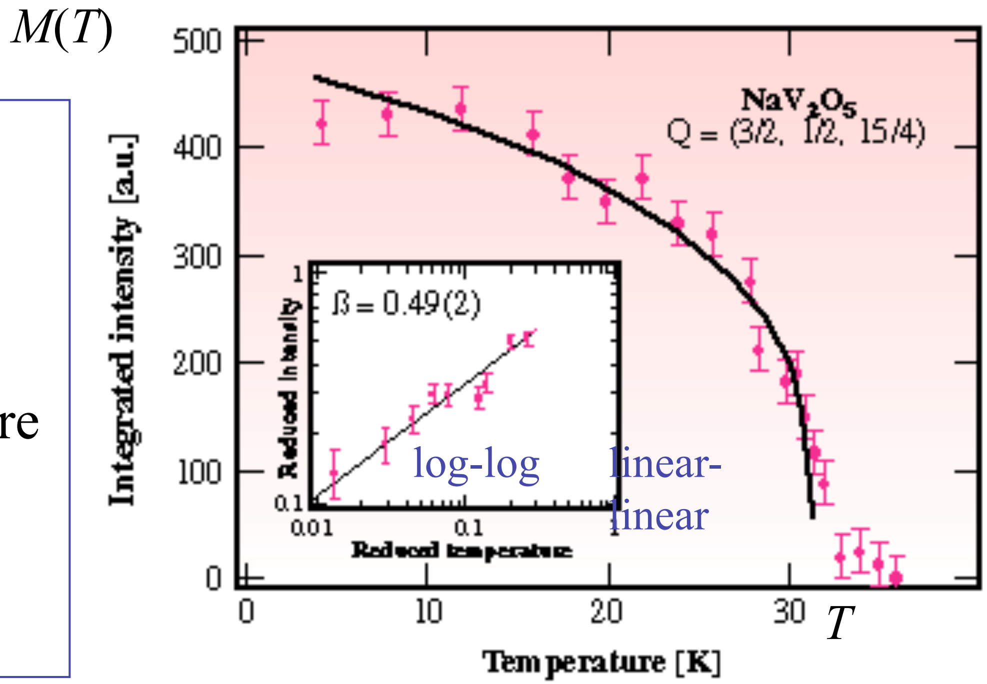
$$t = (T_C - T)/T_C$$

and $m = M/M_0$:

the order parameter scales with temperature

like $m = t^\beta$,

or $\ln m = \beta \ln t$

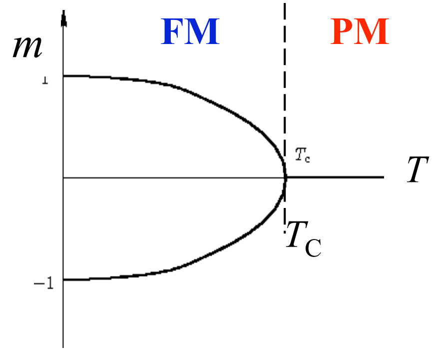


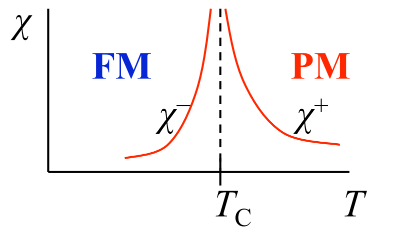
Temperature dependence of magnetisation measured by magnetic scattering of neutrons

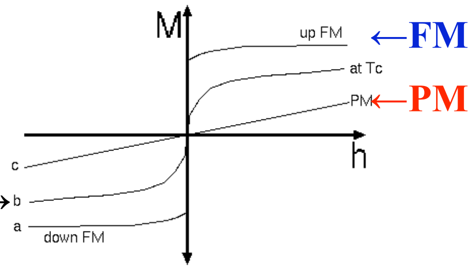
six main critical exponents

critical expon't:

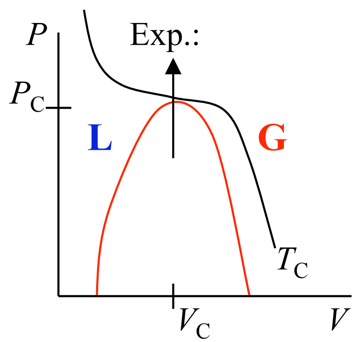
1. Order parameter $m = t^{\beta}$ — β
2. Susceptibility $\chi = |\, t\,|^{-\gamma}$ — γ
3. Critical order param. $M_c = |\, h\,|^{1/\delta}$ — δ
   critical magnetization →
4. Specific heat $C = |\, t\,|^{-\alpha}$ — α
5. Coherence length $\xi = |\, t\,|^{-\nu}$ — ν
6. Correlation function $G = 1/r^{\,d-2+\eta}$ — η

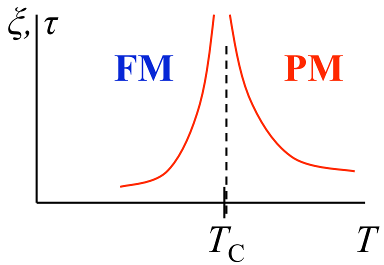






water:





universality

All systems belonging to the same *universality class* have the same critical exponents.

Example: Water near critical point and (1-dim) Magnet

What defines a universality class?

latent heat

Heat a block of ice:

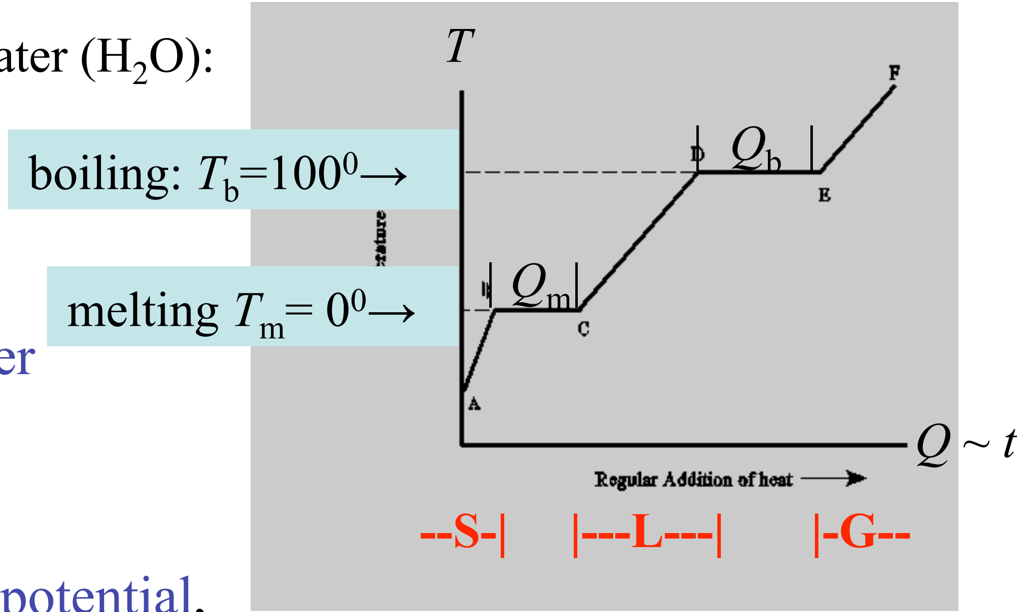
Melting **S** → **L**

Transition: order → short range order

Boiling **L** → **G**

Transition: short range order → disorder

Water (H_2O):



Breaking of bonds requires energy =

latent heat = difference in electrostatic potential,

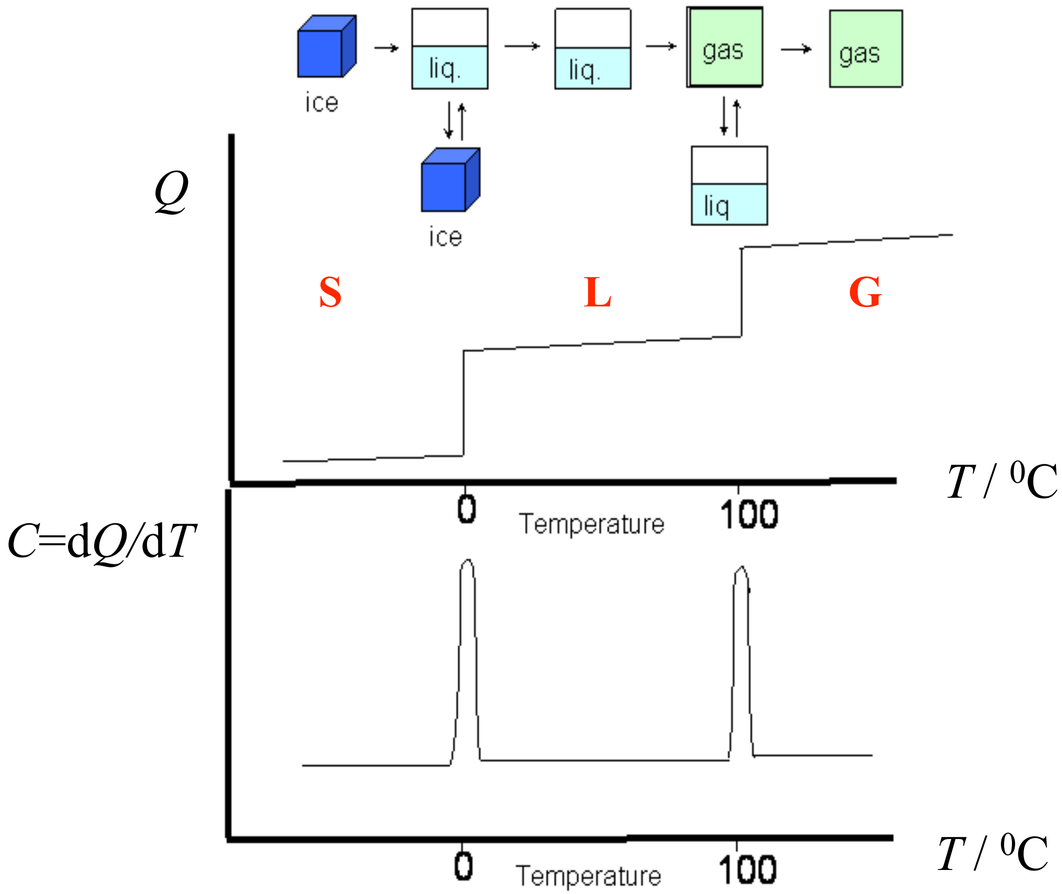
without change in kinetic energy (temperature).

$$P \propto e^{-E/kT} = e^{-E_{kin}/kT} e^{-E_{pot}/kT}$$

At critical temperature T_C:

Addition of heat only changes ratios ice/water or water/vapor, but not the temperature

divergence of heat capacity



1^{st} and 2^{nd} order phase transitions

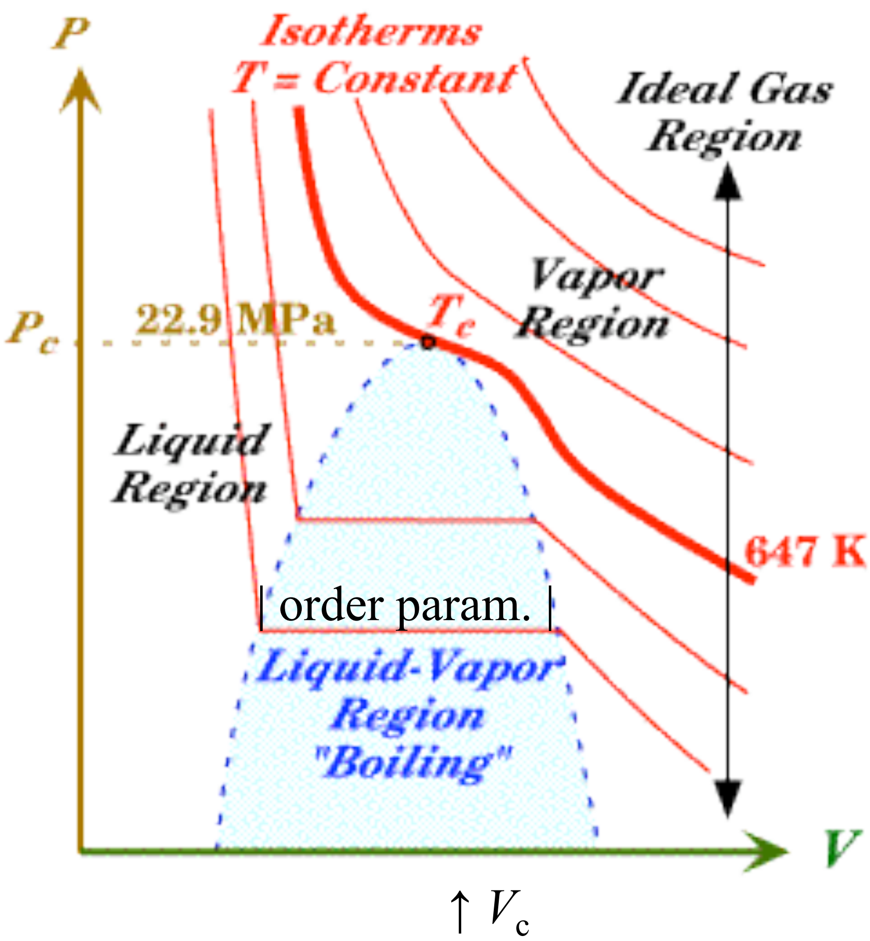
Latent heat: $Q_b = \int_{L\rightarrow G} P\,dV$
= area in P-V diagr.

When latent heat: $Q_b > 0$:
1st-order phase transition.

At the critical point latent heat $Q_b = 0$:
Continuous phase transition
(or 2nd order phase transition)

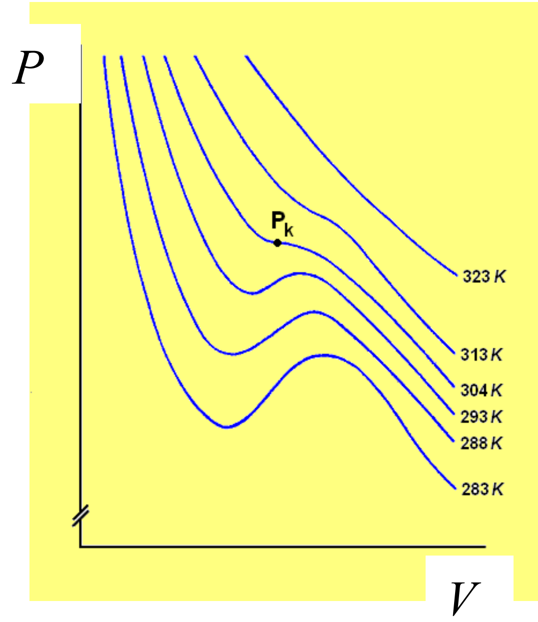
Boiling water:
Order parameter = $\rho_{liquid} - \rho_{gas}$

p-V phase diagram for water (H_2O):



equation of state

Equations of State:
describes reaction to variation of external parameters:



Pressure $P = P(V,T,\dots)$
Magnetization $M = M(B,T,\dots)$

Example:

Ideal gas: $P = RT/V = \rho kT$ Gas equation

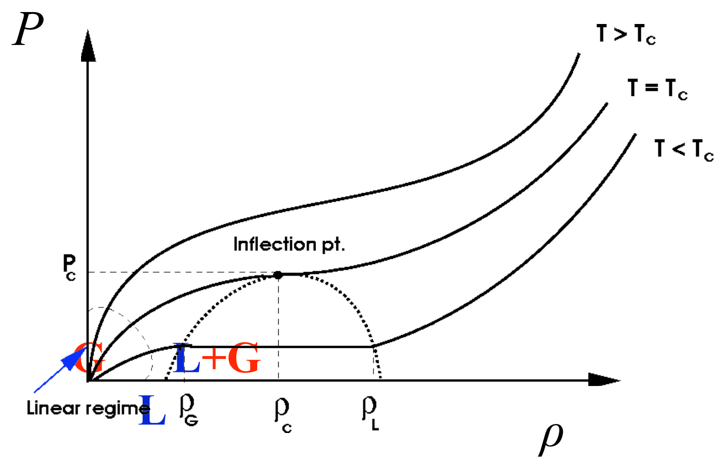
$(\rho = N_A/V,\ R = N_A k)$

Real gas: $(P + a/V^2)(V - b) = RT$

attractive ↑ ↑ repulsive potential

van der Waals-equation

same in P-ρ diagram:



universality of v.d.W.-equation

Bild Yeomans p. 28:

Reduced van der Waals-equation:

$(P/P_C + 3(V_C/V)^2)\,(V/V_C - 1/3) = 8RT_C$

with critical values P_C, V_C, T_C

seems to be universal

even far away from T_C.

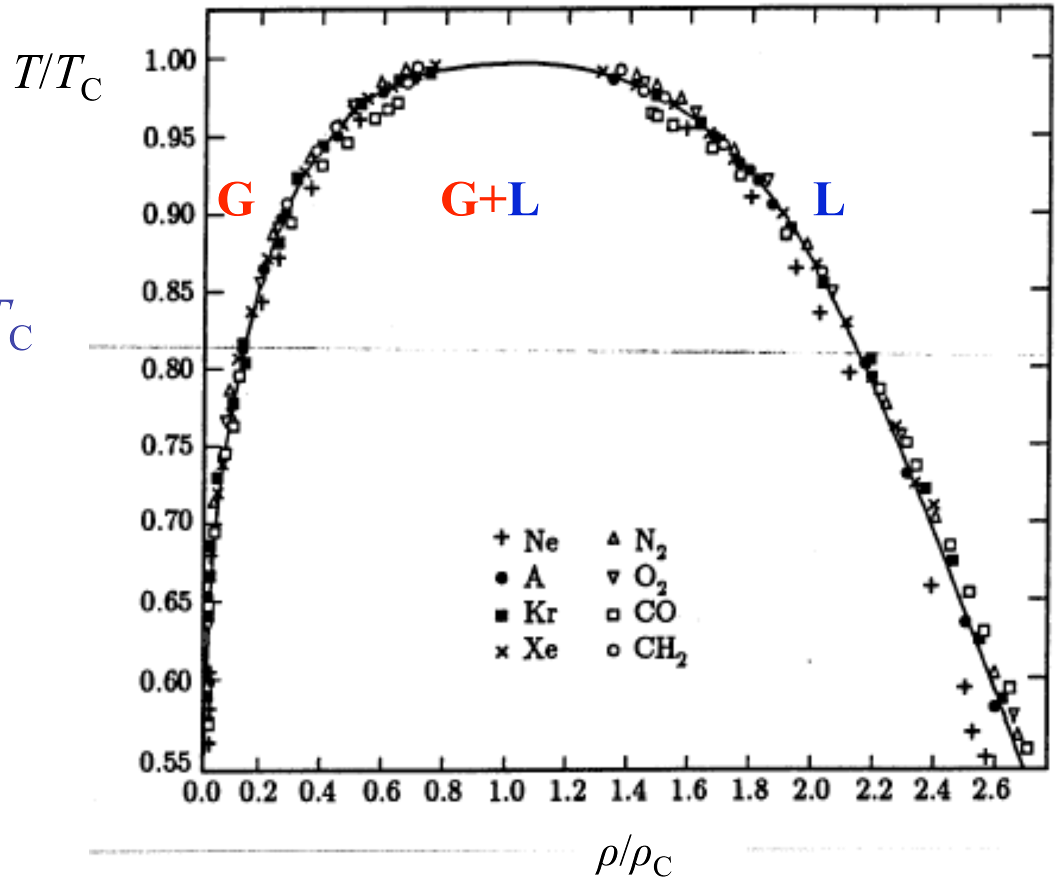


Fig. 4.4.4. Phase boundary in units of reduced temperature and density for eight different molecular fluids near their liquid-gas transitions. Note the universal behavior and the fact that the solid line is $\Delta\phi \propto (T_c - T)^\beta$ with $\beta = 1/3$ rather than the mean-field result $\beta = 1/2$. [E.A. Guggenheim, *J. Chem. Phys.* 13, 253 (1945).]

free energy → everything else

Yeomans p.17:

master plan

Partition function

$$\mathcal{Z}(T,H) = \sum_r e^{-\beta E_r} \qquad (\beta = 1/kT)$$

↓

Free energy

$$\mathcal{F} = -kT \ln \mathcal{Z}$$

↙ ↓ ↘

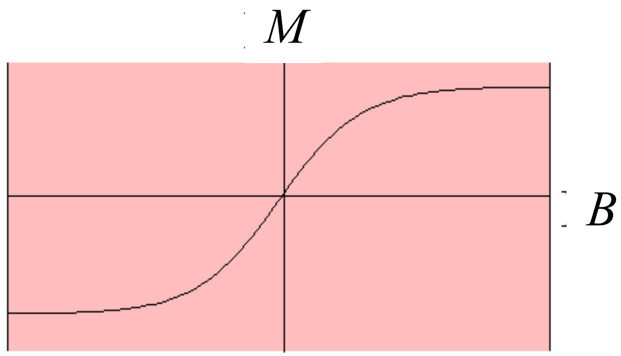
Internal energy	*Entropy*	*Magnetization*
$U = -\frac{\partial \ln \mathcal{Z}}{\partial \beta}$	$S = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_H = (U - \mathcal{F})/T$	$M = -\left(\frac{\partial \mathcal{F}}{\partial H}\right)_T$
↓	↓	↓
Specific heat (constant H)	*Specific heat (constant $X = H, M$)*	*Isothermal susceptibility*
$C_H = \left(\frac{\partial U}{\partial T}\right)_H$	$C_X = T\left(\frac{\partial S}{\partial T}\right)_X$	$\chi_T = \left(\frac{\partial M}{\partial H}\right)_T$

example: paramagnetism

spin ½:

energy/molecule $E_\pm = \pm\mu B$

PM:



partition function for N molecules:

$$Z = \left(\sum_r e^{-\beta E_r}\right)^N = (e^{-\beta E_+} + e^{-\beta E_-})^N \qquad \beta = 1/kT$$

magnetisation $$< M > = NkT\frac{1}{Z}\frac{\partial Z}{\partial B} = M_0 \frac{e^{-\beta E_+} - e^{-\beta E_-}}{e^{-\beta E_+} + e^{-\beta E_-}}$$

$$< M > = N\mu \tanh\left(\frac{\mu B}{kT}\right) \approx \frac{\mu B}{kT}$$

Saturation magnetis. $M_0 = N\mu$

Susceptibility $\chi = \partial M/\partial B \approx N\mu^2/kT$: $\chi \sim 1/T$

= Curie Law, for $kT >> \mu B$

3. Landau model

Landau 1930 (Landau-Lifschitz 5: Statistical Physics ch. XIV)

'Landau' free energy of ferromagnet $F = F(m)$

with magnetization $m = \langle M \rangle / M_0$ (mean field approx.)

Taylor-expanded about $m = 0$:

$$F = F_0(T) + (\tfrac{1}{2}a'\, m^2 + \tfrac{1}{4}\lambda\, m^4)V$$

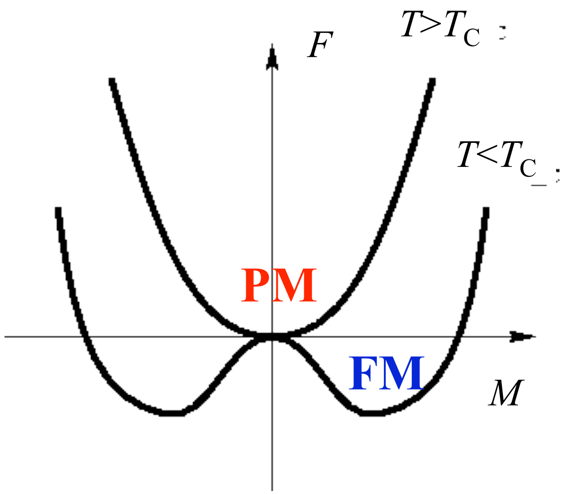
(only even powers of m, $\lambda > 0$)

a' changes sign at $T = T_C$: $a' = a \cdot (T - T_C)$:

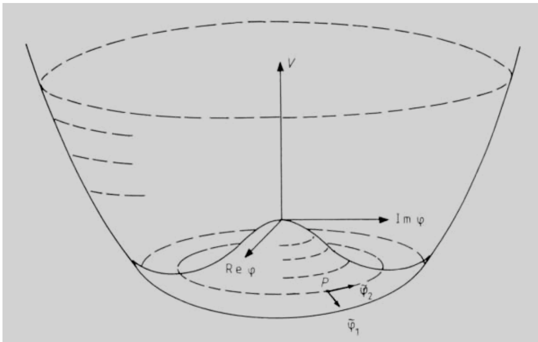
Free energy density $f = (F - F_0)/V$ then is:

$$f = \tfrac{1}{2}a(T - T_C)\, m^2 + \tfrac{1}{4}\lambda\, m^4$$

1-dim magnet:



2-dim magnet:



Spontaneous magnetization in zero external field

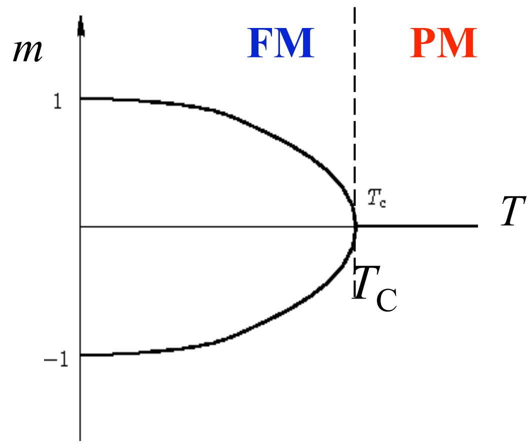
Landau: $f = \frac{1}{2}a(T-T_C)\, m^2 + \frac{1}{4}\lambda m^4$

At equilibrium → minimum of free energy:

$$\partial f/\partial m = a(T-T_C)\, m + \lambda m^3 = 0$$

and

$$a(T-T_C) + \lambda m^2 = 0$$



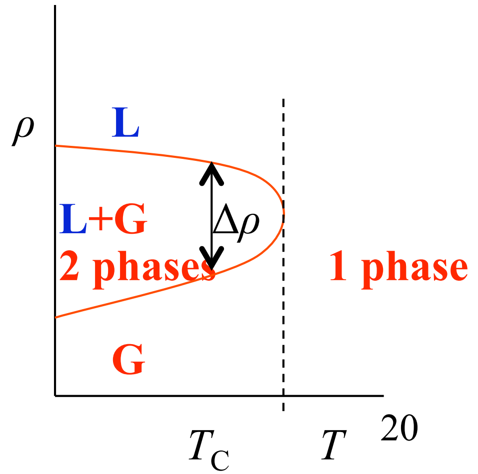
for $T \geq T_C$: Magnetization $m = 0$: **PM**

for $T < T_C$: Magnetization $m = \pm(a/\lambda)^{1/2}(T_C-T)^{1/2}$ **FM**

first critical exponent $\beta = \frac{1}{2}$

Same result for order parameter of v.d.W. gas:

$$\rho_L - \rho_G \sim (T_C - T)^{1/2}.$$



Magnetization in external field

Magnetic energy in external magnetic field h:

$$f = \tfrac{1}{2}a(T - T_C)\, m^2 + \tfrac{1}{4}\lambda m^4 - hm$$

At equilibrium: from

$$\partial f/\partial m = a(T - T_C)\, m + \lambda m^3 - h = 0$$

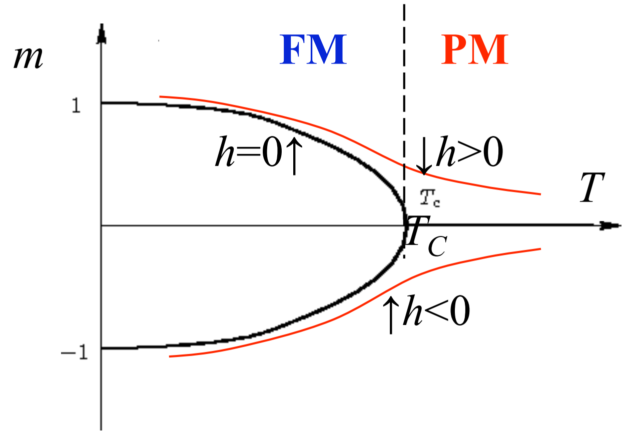
follows magnetization $\pm m(T)$, see Fig.,

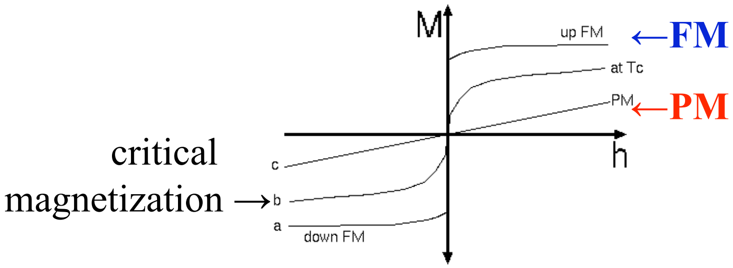
in particular:

critical magnetization at $T = T_C$:

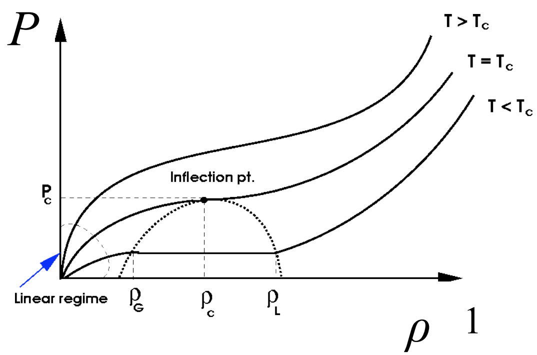
$$h = \lambda m^3$$

third critical exponent $\delta = 3$

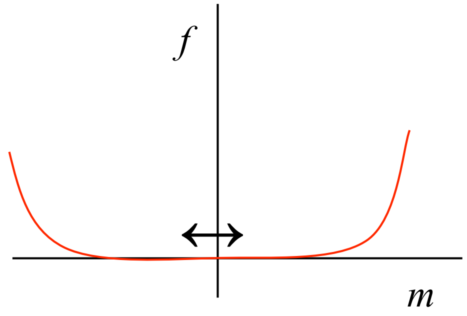




Same result for critical isotherm parameter of v.d.W. gas:



Magnetic susceptibility



Susceptibility $\chi = \partial m / \partial h$ diverges at $T = T_C$

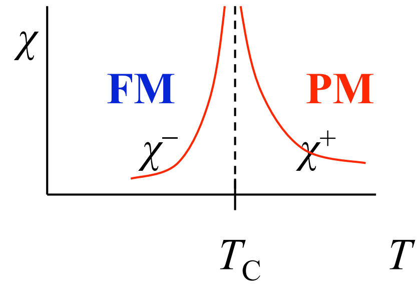
Reason: Free energy has flat bottom at $T = T_C$:

above T_C: $\chi^+ = [a(T - T_C)]^{-1}$ **PM**

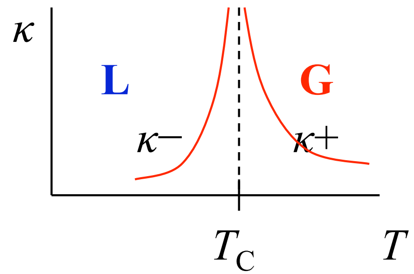
below T_C: $\chi^- = [2a(T_C - T)]^{-1} = \frac{1}{2}\chi^+$ **FM**

Curie-Weiss law

second critical exponent $\gamma = 1$



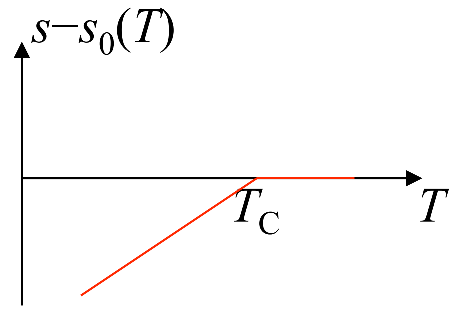
Same result for v.d.W.-compressibility κ^- and κ^+:



Specific heat in zero field

Entropy density

$s - s_0(T) = -\partial f / \partial T$

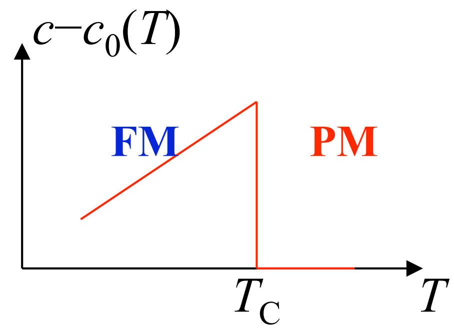


Specific heat

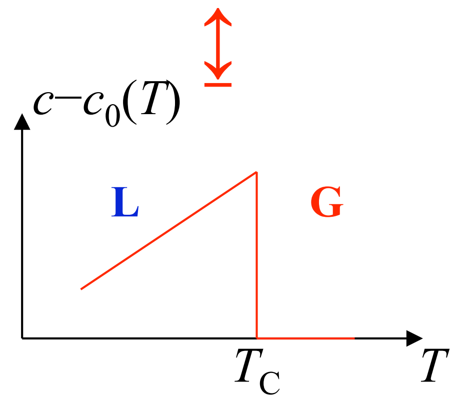
$c - c_0(T) = -\partial s / \partial T$

Result: Specific heat makes a jump at T_C:

<u>fourth critical exponent $\alpha = 0$</u>

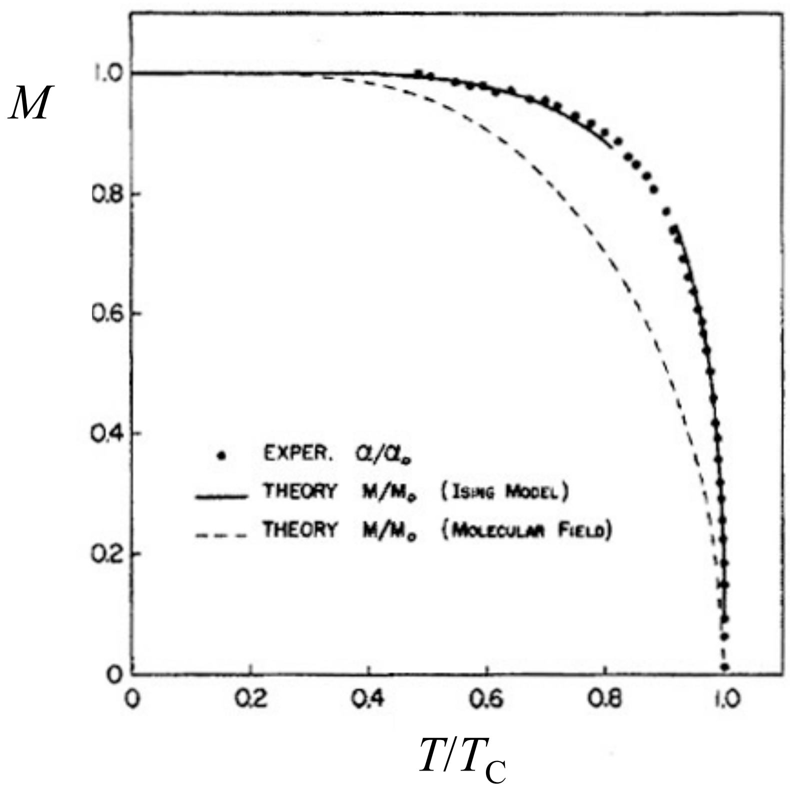


Same result as for specific heat of v.d.W. gas:



Compare with experiment

Magnetization: $M \sim (T_C - T)^{\beta}$



specific heat has a small $\alpha > 0$

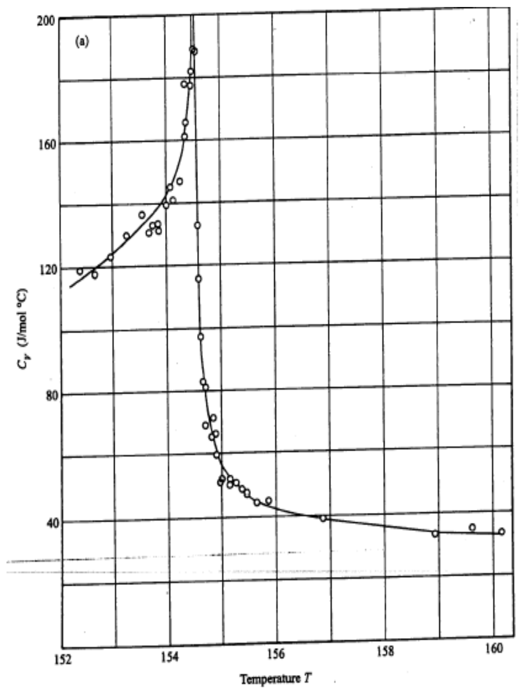


Figure 5.25 (a) Temperature dependence of C_V of oxygen at $\rho \sim \rho_c = 0.408$ g/cm^3; (b) (opposite) dependence of C_V of oxygen on $\ln|T - T_c|$.

	c	M	χ	M_c
	α	β	γ	δ
Landau:	0	1/2	1	3
Exp.	0.1	0.34	1.35	4.2

4. Ginzburg-Landau theory of superconductivity

Non-uniform superconductor (mixed phases, Meissner effect, etc.):
order parameter = cooper pair wavefct. is position dependent: $\psi = \psi(\boldsymbol{r})$

like in elasticity theory:
energy penalty for deviations from homogeneity is ~ $|\nabla\psi|^2$.
Free energy:

$$F_s = F_n + \int_V \left((\hbar^2/2m^*)\, |\nabla\psi|^2 + \tfrac{1}{2}\mu^2 \cdot (T - T_c)\, |\psi|^2 + \tfrac{1}{4}\lambda |\psi|^4 \right) \mathrm{d}V$$

$$E_{kin} \quad + \quad E_{pot}$$

(Ginzburg-Landau, 1950)

superconductor in magnetic field

$\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{r}) = \nabla\times\boldsymbol{A}(\boldsymbol{r})$, with vector potential $\boldsymbol{A}$,

$\boldsymbol{A}$ changes momentum $m\boldsymbol{v}$ of a particle to

$$\boldsymbol{p} = m\boldsymbol{v} + e\boldsymbol{A}$$

but does not change its energy

$$E = (m\boldsymbol{v}^2)/2m = (\boldsymbol{p}-e\boldsymbol{A})^2/2m$$

so for $\boldsymbol{B} \neq 0$

$$F_s = F_n + \int_V (|-i\hbar\nabla\psi_s - e^*\boldsymbol{A}\psi|^2 /2m^* + ½\mu^2\cdot(T-T_c)\,|\psi_s|^2 + ¼\lambda|\psi_s|^4 + \boldsymbol{B}^2/2\mu_0 - \boldsymbol{B}\cdot\boldsymbol{M})\,\mathrm{d}V$$

T_{sc} $+ V_{sc}$ $+ E_{\mathrm{field}}$ $+ E_{\mathrm{magn}}$

$m^* = 2m_e$, $e^* = 2e$

Lit.: C.P. Poole et al.: Superconductivity, ch.5, Academic Press 1995

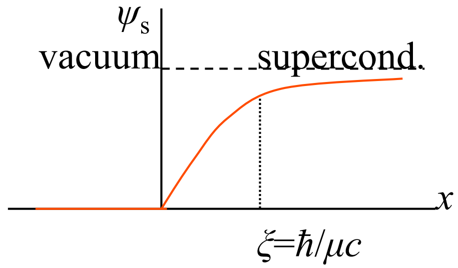
two scales in superconductivity

Mean field theory of superconductivity (Ginzburg-Landau):

Superconductor has 2 characteristic scales:

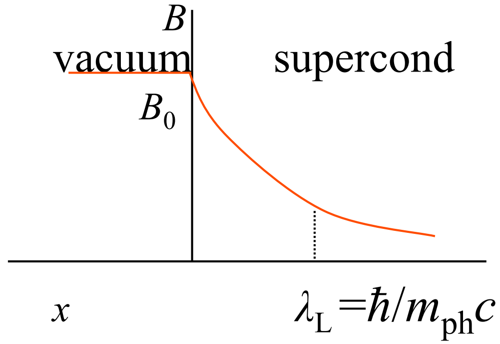
1. of the order parameter = superconducting condensate ψ:

   coherence length $\xi = \hbar/\mu c$ of the condensate



2. of the magnetic field, via the Meissner effect:

   London penetration depth $\lambda_L = \hbar/(evc)$



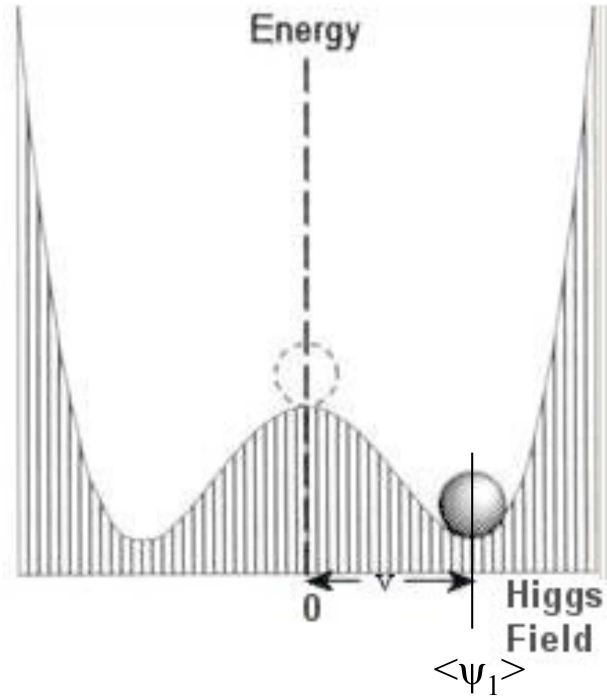
Superconductivity and Standard Model

Wir vergleichen :

Superconductor, Meissner - Ochsenfeld effect:
Gintzburg - Landau Lagrange density

$$L_s = L_n + \frac{\hbar^2}{2m*}\left|\mathbf{D}\psi_s\right|^2 - ½\mu^2\left|\psi_s\right|^2 - ¼\lambda\left|\psi_s\right|^4 - \frac{B^2}{2\mu_0} - \boldsymbol{B}\cdot\boldsymbol{M}$$

with covariant derivative $\mathbf{D} = \vec{\nabla} - ie*\boldsymbol{A}$
Cooper condensate : $m* = 2m, \quad e* = 2e, \ \psi_s = \psi_1 + \mathrm{i}\psi_2,$
mit $\mu^2 \propto (T_C - T)$ $\qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}$



Standard model, Higgs mechanism :
Weinberg - Salam Lagrange density

$$L = (\mathrm{D}_\mu\Phi)^*(\mathrm{D}_\mu\Phi) - ½\mu^2(\Phi^*\Phi) - ¼\lambda(\Phi^*\Phi)^2 - \frac{1}{4}W_{\mu\nu}W_{\mu\nu} + \ldots$$

with covariant derivative $\mathrm{D}_\mu = \partial_\mu + igA_\mu\cdot\tau + \frac{1}{2}ig'B_\mu$

field tensors : $\boldsymbol{W}_{\mu\nu} = \partial_\mu\boldsymbol{A}_\nu - \partial_\nu\boldsymbol{A}_\mu - g\boldsymbol{A}_\mu\times\boldsymbol{A}_\nu$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

mit $\mu^2 \propto (T_C - T)$

Comparison of coefficients:

	G.-L.: $U_{el\text{-}mag}(1)$	W.-S.: $SU_L(2)\times U_Y(1)$
order parameter:	super-conducting condensate $\psi = \psi_1 + i\psi_2$	Higgs doublet $\Phi = \begin{pmatrix} \Phi_1 + i\Phi_2 \\ \Phi_3 + i\Phi_4 \end{pmatrix}$
boson mass generation by Higgs field:	Meissner effect $m_{ph} = e <\psi_1>$	Higgs mechanism $m_W = g <\Phi_3>$
Compton wavelength λ of interacting boson:	London penetration depth $\lambda_L = \hbar/(m_{ph}c)$	range of weak interaction $\lambda_W = \hbar/(m_W c)$
Compton wavelength λ of Higgs:	coherence length $\xi = \hbar/(\mu c)$	"coherence length" $\lambda_H = \hbar/(m_H c)$

N.B.: die maximale Reichweite einer WW durch virtuellen Austausch ist
$R = c\Delta t = \hbar c/E_0 = \hbar/mc \equiv \lambda_{Compton}$

Dasselbe im Detail:

Im Folgenden zeigen wir:

1. das Entstehen von Goldstone Bosonen
   anhand der Magnonen-Anregung in einem Ferromagneten im Landau Modell (spontane Brechung einer *globalen* Symmetrie)

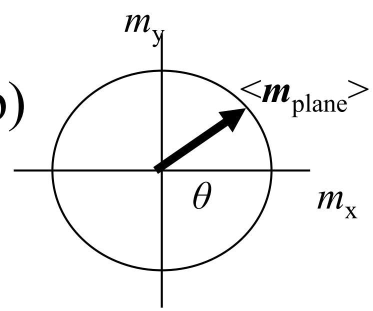
2. den Higgs-Mechanismus, dh. das Verschwinden des Goldstone Bosons und die Entstehung der Masse des Eichbosons
   anhand eines Supraleiters im Landau-Gintzburg Modell
   (spontane Brechung einer *lokalen* Eichsymmetrie)

Goldstone's theorem for Landau magnet

Goldstone's theorem:

Each spontaneous breaking of a continuous symmetry creates a massless particle (i.e. an excitation without an energy gap)

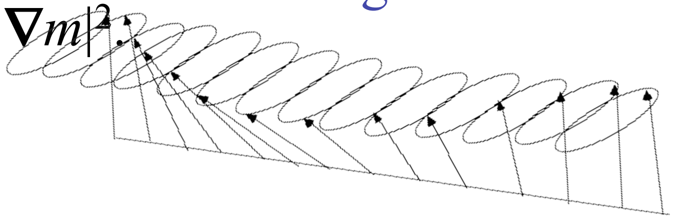
= Goldstone Boson



Simple example: Landau ferromagnet: like in elasticity theory:

energy penalty for deviations from homogeneity is ~ $|\nabla m|^2$.

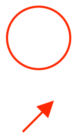
magnon:



$$f = \nabla m^2 + \tfrac{1}{2}\mu^2 (T - T_C)|m|^2 + \tfrac{1}{4}\lambda |m|^4$$

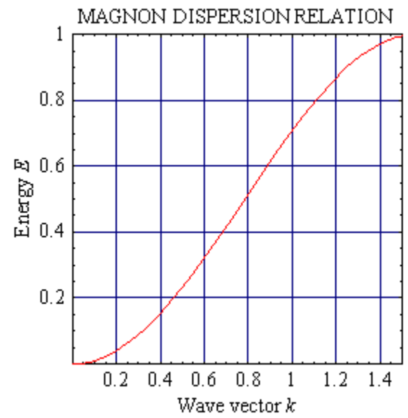
solution above T_C is rotationally symmetric:

solution below T_C is cylindrically symmetric:



Goldstone mode belonging to broken symmetry = magnon

magnon dispersion relation:



Magnon = Goldstone of broken PM→FM symm.

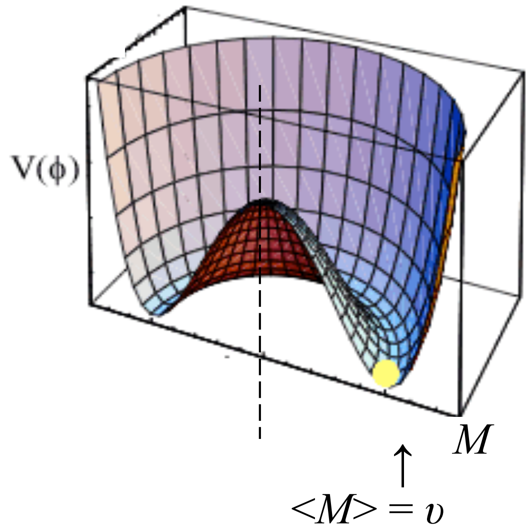
$$L = \tfrac{1}{2}\left|\vec{\nabla} M\right|^2 - \tfrac{1}{2}\mu^2\left|M\right|^2 - \tfrac{1}{2}\lambda\left|M\right|^4$$

$T < T_C$:

Fluktuationen um neues Minimum $\upsilon = \langle M \rangle$:

Fluktuation der Amplitude $\chi(x) = \Delta M$ und Fluktuation der Phase φ :

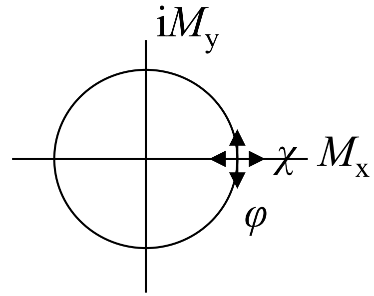
$$M(x) = (\upsilon + \chi(x))\,\mathrm{e}^{\mathrm{i}\varphi(x)/\upsilon}$$



Kettenregel : $\tfrac{1}{2}\left|\vec{\nabla} M\right|^2 = \tfrac{1}{2}\left|\vec{\nabla}\chi\,\mathrm{e}^{\mathrm{i}\varphi(x)/\upsilon} + \mathrm{i}\vec{\nabla}\varphi(x)/\upsilon\,(\upsilon + \chi)\,\mathrm{e}^{\mathrm{i}\varphi(x)/\upsilon}\right|^2$

Fluktuation $\chi << \upsilon$: $\approx \tfrac{1}{2}\left|\vec{\nabla}\chi + \mathrm{i}\vec{\nabla}\varphi\right|^2$

$= \tfrac{1}{2}\left|\vec{\nabla}\chi\right|^2 + \tfrac{1}{2}\left|\vec{\nabla}\varphi(x)\right|^2$ (wegen $\left|a + \mathrm{i}b\right|^2 = a^2 + b^2$)



und : $-\tfrac{1}{2}\mu^2\left|M\right|^2 = -\tfrac{1}{2}\mu^2\left|\upsilon + \chi\right|^2 = \text{const.} - \tfrac{1}{2}\mu^2\left|\chi\right|^2$

(der in χ lineare Term $-\tfrac{1}{2}\mu^2\upsilon\chi$ verschwindet im Minimum)

eingesetzt in L ergibt :

$L = \text{const.}$ = von x unabhängige Terme

$+\tfrac{1}{2}\left|\vec{\nabla}\chi\right|^2 - \tfrac{1}{2}\mu^2\left|\chi\right|^2$ = "massive" Anregung χ um neuen Grundzustand $\langle M \rangle = \upsilon$ längs $\boldsymbol{M}$

$+\tfrac{1}{2}\left|\vec{\nabla}\varphi\right|^2$ = Goldstone φ ohne Massenterm = **Magnon ohne Energielücke**

+ ... = WW. höherer Ordnung

spontaneous symm.breaking in superconductor

Gauge invariance requires interaction with massless field A_μ.

However, in a superconductor, the photon A_μ becomes massive.

Still: Ginzburg-Landau model is gauge invariant (Dr.-thesis Ginzburg ~ 1950)

The reason is what is now called the Higgs mechanism:

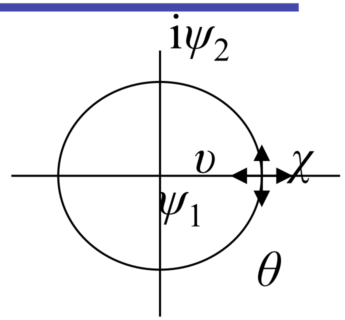
When a scalar, gauge invariant field ψ
suffers a spontaneous symmetry breaking,
then the vectorfield A_μ can become massive,
without losing its gauge invariance,
while at the same time the Goldstone disappears.

Ginzburg-Landau superconductor: Cooper pairs = Higgs field ψ:

$$L_s = L_n + |\nabla\psi - i2e\boldsymbol{A}\psi|^2 - \tfrac{1}{2}\mu^2|\psi|^2 - \tfrac{1}{4}\lambda|\psi|^4 - \boldsymbol{B}^2/\mu_0{}^* + \boldsymbol{B}\cdot\boldsymbol{M}$$

with charge of Cooper pairs $e^*=2e$.

Higgs-mechanism for superconductor



As before: Fluctuations of $\psi(x)$ about $<\psi> = \upsilon$ at "bottom of bottle"

$$\psi(x) = (\upsilon + \chi(x)) \exp(i\theta(x)/\upsilon)$$

Then $|\nabla\psi - ie^*\boldsymbol{A}\psi|^2 = |\nabla\chi + i(\upsilon+\chi)(\nabla\theta/\upsilon - e^*\boldsymbol{A})|^2$, with $\chi << \upsilon$:

if we choose gauge to $\boldsymbol{A}=\boldsymbol{A}'+\nabla\theta/e^*\upsilon$, this becomes $\approx (\nabla\chi)^2 - \upsilon^2 e^{*2}\boldsymbol{A}^2$,

and the massless Goldstone term $(\nabla\theta)^2$ disappears,

and the photon $\boldsymbol{A}$ becomes massive (but remains gauge invariant):

$L_s = \text{const.} + (\hbar^2/2m^*)\,(\nabla\chi)^2 - \frac{1}{2}\mu^2|\chi|^2$	= "Higgs" with mass μ
$-m_{ph}^2\boldsymbol{A}^2$	= heavy photon with mass $m_{ph}=\upsilon e^* = (a\cdot(T-T_C)/\lambda)^{1/2}\, 2e$
$-\boldsymbol{B}^2/2\mu_0 + \boldsymbol{B}\cdot\boldsymbol{M}$	= field terms as before
	+ some residual terms

The coherence length found before turns out to be $\xi = 1/\mu$ ($\hbar=c=1$),

or $\xi = \hbar/\mu c$ = Compton wave length of the Higgs of mass $\mu = (4ma\cdot(T_C-T))^{1/2}$,

and the London penetration depth $\lambda_L = 1/m_{ph}$, or $\lambda_L = \hbar/m_{ph}c$

= Compton wave length of the heavy photon

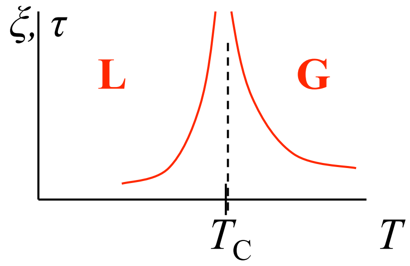
N.B.: number of degrees of freedom remains the same

5. Fluctuations

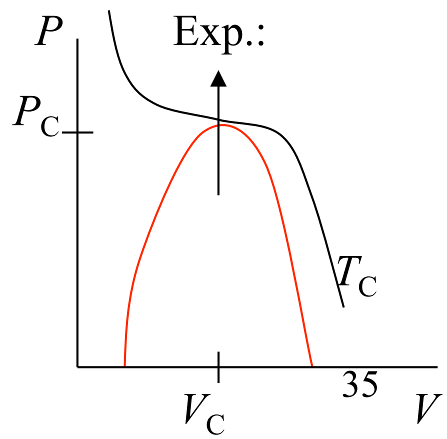
Ginzburg-Landau and Weinberg-Salam are mean-field theories, which is sufficient.

But: in general time-dependent fluctuations must be included.

Correlation length $\xi \sim$ mean size of a region of same density
Correlation time $\tau \sim$ mean time of existence of such a region



When $T \rightarrow T_C$, then density fluctuations on all length scales and all time scales



Divergence: $\xi \rightarrow \infty$, $\tau \rightarrow \infty$

MOVIE: CRITICAL SCATTERING

Critical fluctuations

Fluctuations are intimately linked to the susceptibilities

In a magnet: mean square fluctuations of magnetization:

$$\langle \boldsymbol{M}^2 \rangle - \langle \boldsymbol{M} \rangle^2 = kT \cdot \chi,$$ with magnetic susceptibility χ.

At the Curie temperature: $T = T_C$ the critical magnetic fluctuations diverge like the static susceptibility χ,
therefore critical exponents cannot be all independent of each other.

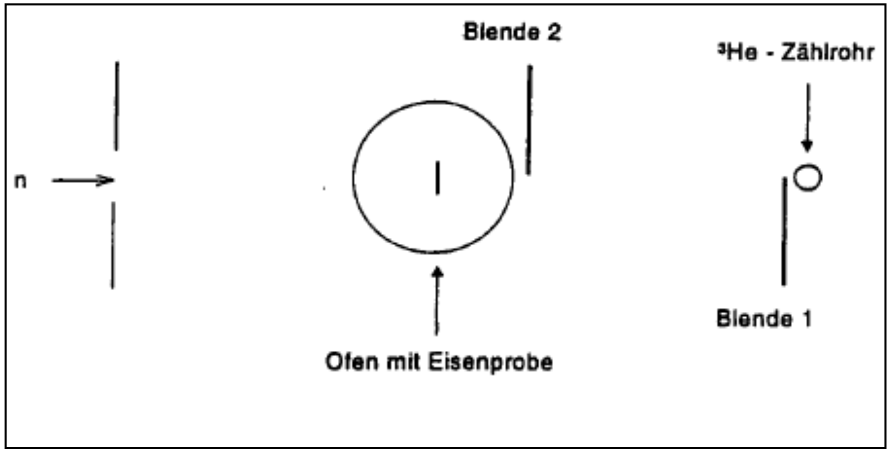
(same in liquid: density fluctuations are linked to compressibility κ)

(cf: dissipation-fluctuation theorem, Nyquist theorem)

Measurement of magnetic critical opalescence

Staatsexamens-Arbeit N. Thake 1999

Abbildung 5-6 Versuchsaufbau zur Messung der kritischen Streuung



$$\chi^{+}(T) \sim (T - T_C)^{-\gamma}$$
$$\chi^{-}(T) = \tfrac{1}{2}\,\chi^{+}(T)$$

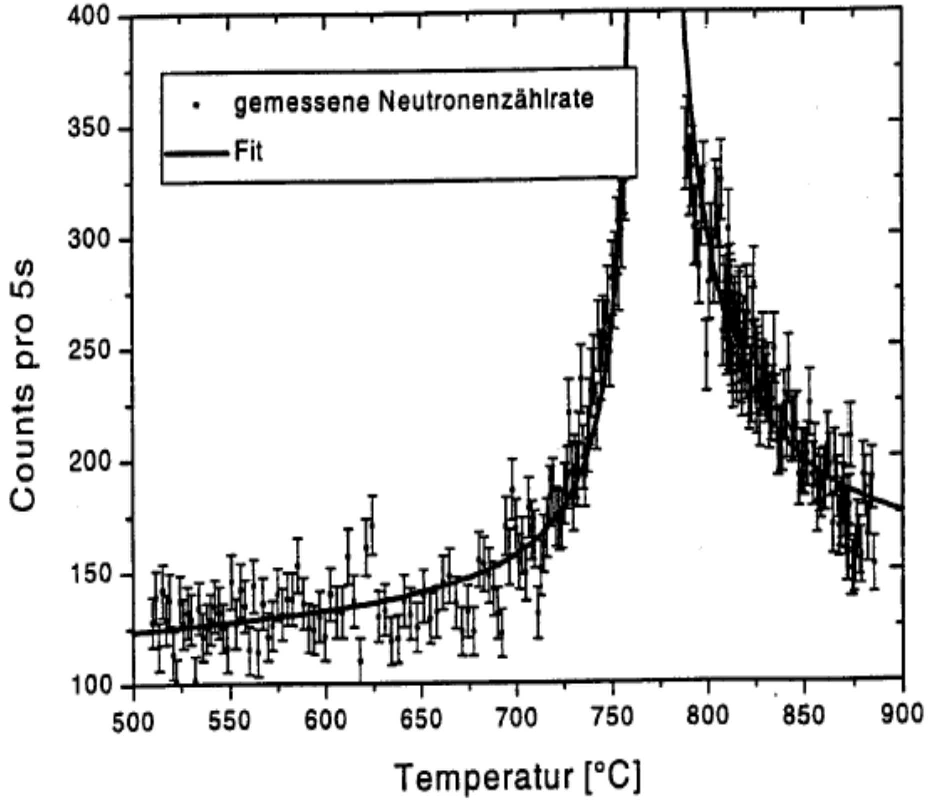


Abbildung 5-8 Messung der kritischen Streuung mit zugehörigem Fit

Correlation function near the critical point

For many systems the spatial correlation function decays with distance r like:

$G(r) \sim \exp(-r/\xi)/r^{\mathrm{n}}$

Near the critical point the correlation length ξ diverges like

$\xi \sim |T - T_{\mathrm{C}}|^{-v}$,

and the correlation function becomes

$G(r) \sim 1/r^{d+2-\eta}$,

with dimension d, and with two further critical exponents v and η.

only 2 independent critical exponents

Empirically:

Only 2 of the 6 critical exponents are independent,
since the following 4 empirical relations hold:

$$\alpha + 2\beta + \gamma = 2$$

$$\alpha + \beta(1+\delta) = 2$$

$$\gamma = (2-\eta)\nu$$

$$d\nu = 2-\alpha$$

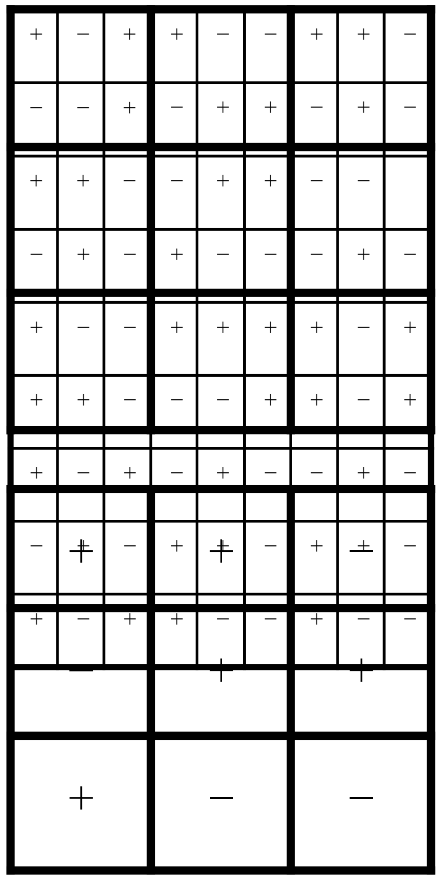
What is the deeper reason for this?

Scale invariance and renormalization

Mean field: Averaging over the fluctuations is not permitted because fluctuation amplitudes diverge at the critical point.

Way out: successive averaging, separately for each scale, starting with a small length scale L << coherence length ξ

(when working in real space).



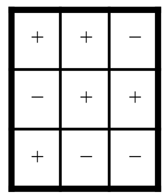
Example for d=2 dimensional ('block-spin') iteration process:

Divide systems in blocks of volume $L^d = 3^2 = 9$ cells.

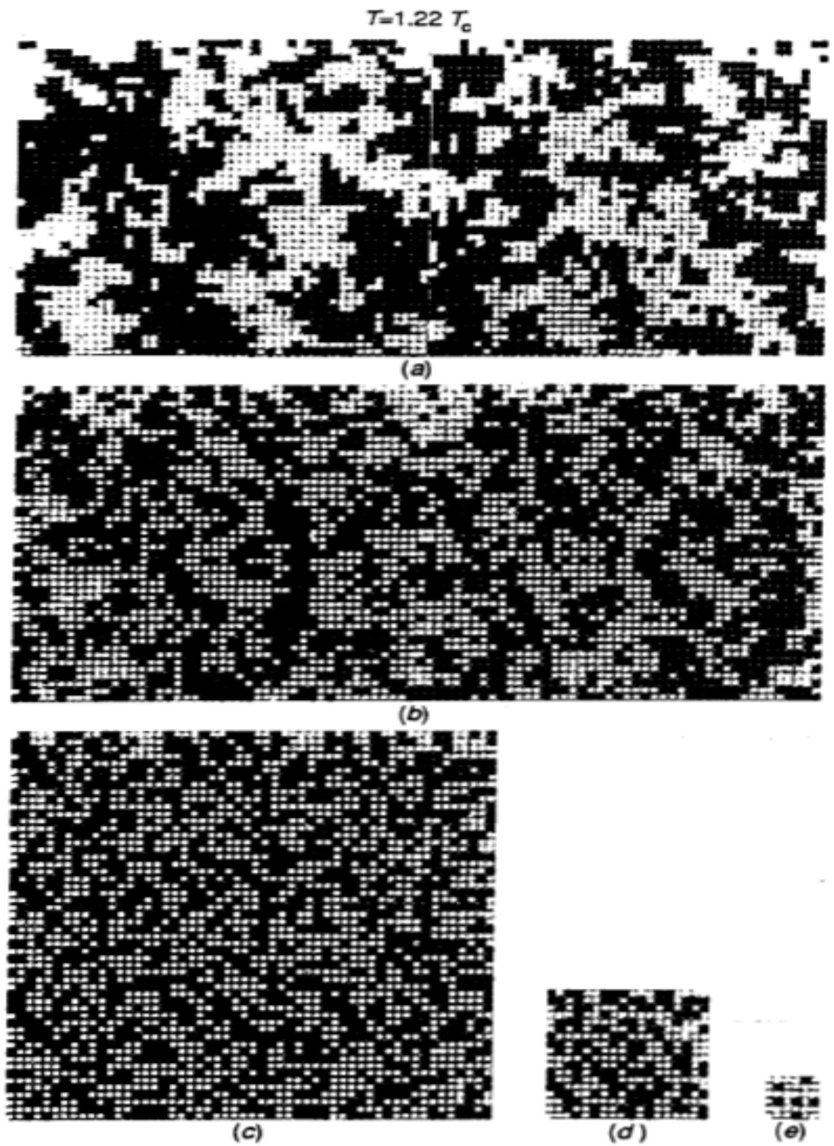
1. Take a majority vote in each block.
2. Combine the cells in a block and assign the majority vote to the cell.
3. Shrink new cells to the size of the original cells and renumber them. Number of configurations shrinks from 2^9=512 to 2^1=2.
4. 'Renormalize' the interaction $\hat{H}$ between the averaged elements such that the new partition function stays the same:

$$Z_{N'} = \sum_{2\text{^}N'\ \text{config}'} e^{-\beta\hat{H}'} = \sum_{2\text{^}N\ \text{config}'} e^{-\beta\hat{H}} = Z_N\,,$$

so that the physics remains the same (scale invariance). Go to 1.



Block-spin operation in 2-dim.



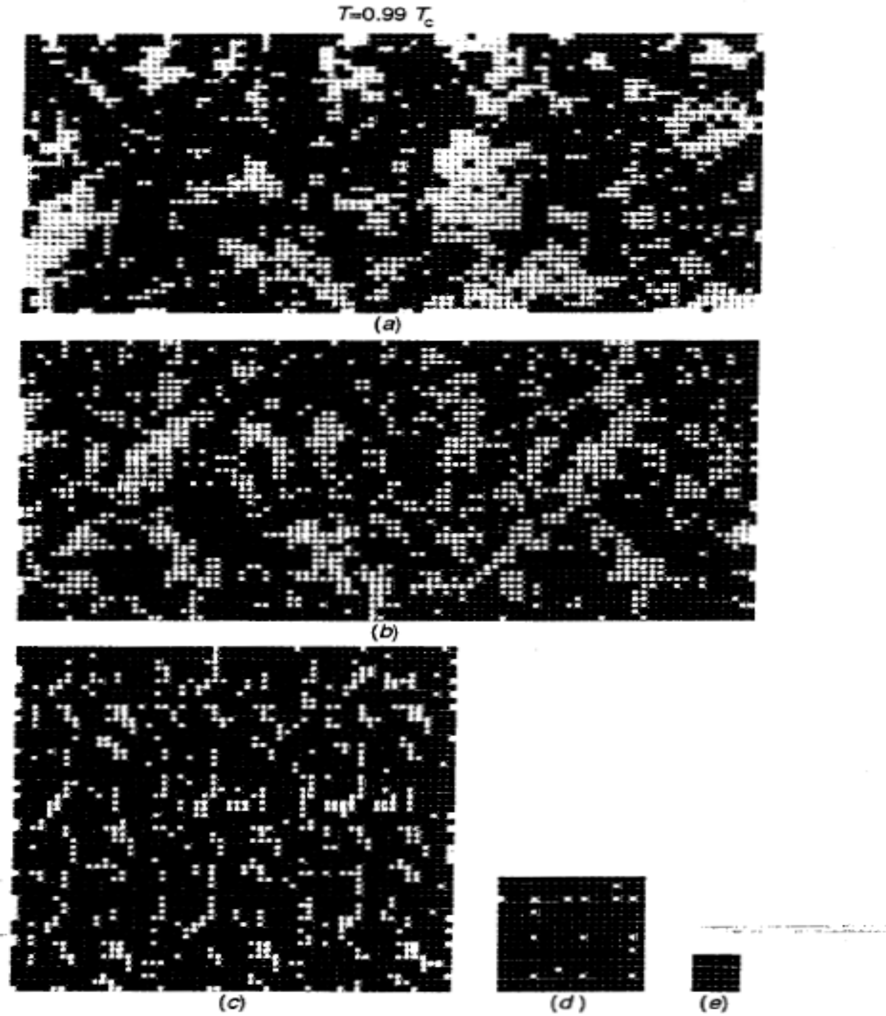


Fig. 1.10. As Fig. 1.8 but with a starting temperature $T = 0.99T_c$. Fluctuations relative to the ordered state are suppressed by the change in length scale and the system flows towards zero temperature. After Wilson, K. G. (1979). *Scientific American*, **241**, 140.

Block-spin operation in 2-dim.: $T = T_C$

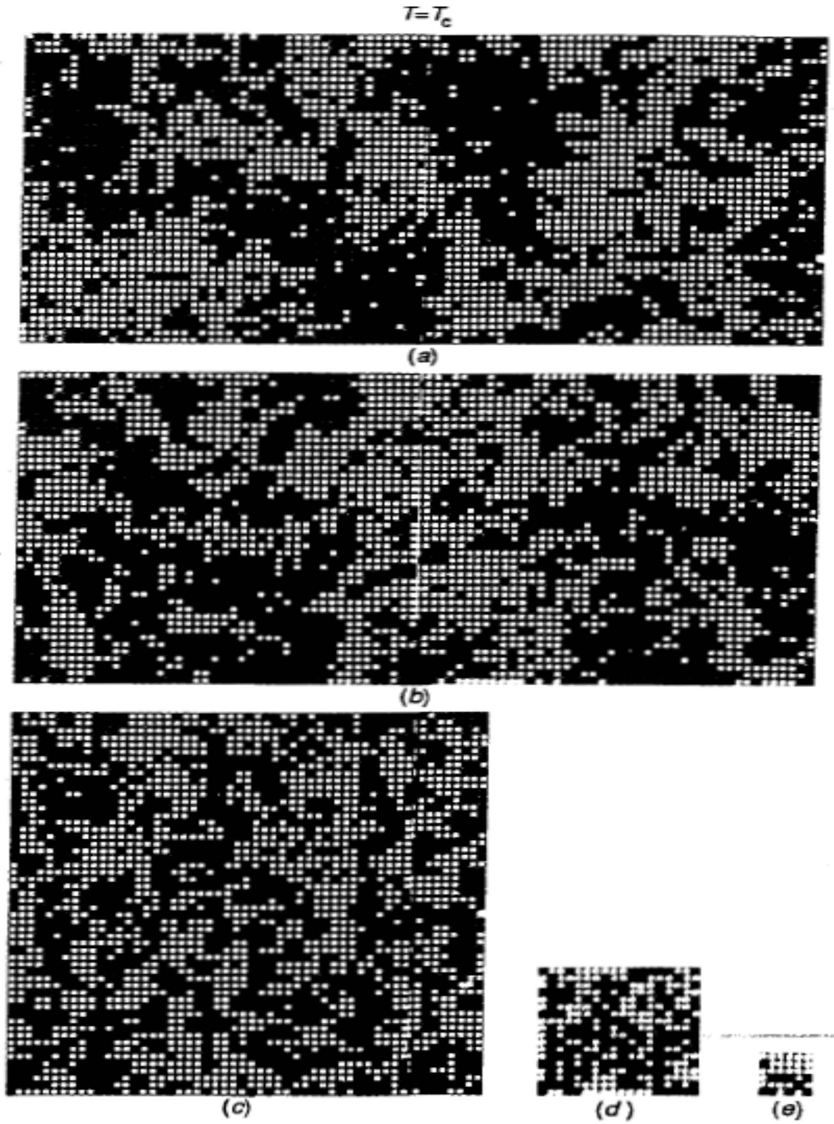


Fig. 1.9. As Fig. 1.8 but with a starting temperature $T = T_c$. Because the correlation length is initially infinite there is no change in the ordered state under iteration of the renormalization group and the system remains at the critical temperature. After Wilson, K. G. (1979). *Scientific American*, **241**, 140.

Transformation of reduced temperature and field

At each iteration step:
coherence length shrinks from ξ to $\xi' = \xi / L$,
that is temperature T moves away from T_C,
either to higher $T \to \infty$ or to lower $T \to 0$ temperatures:

Under an iteration the reduced temperature $t = |(T - T_C)/T_C|$ changes
from t to $t' = g(L)\, t$, the function $g(L)$ is to be determined:
Upon two iterations, successive shrinking is by L_1, then by L_2, in total by L_1L_2.
Reduced temperature changes to $t' = g(L_2)\, g(L_1)\, t = g(L_1L_2)\, t$.
A function with the property $g(L_2)\, g(L_1) = g(L_1L_2)$ necessarily has the form $g(L) = L^{y}$,
Check: $L_1^{y} L_2^{y} = (L_1L_2)^{y}$.

Hence the reduced temperature t transforms as: $t' = L^{y}\, t$ with exponent $y>0$.
Same argument for magnetic field: it increases when coherence length shrinks:
i.e. reduced field h transforms as: $h' = L^{x}\, h$ with exponent $x>0$.

Critical exponent relations from scaling

Order parameter magnetization:

$m = -\partial f\,(t, h),\ /\partial h|_{h\to 0}$

$\quad = L^{-d} L^{x}\,\partial f\,(L^{y} t, L^{x} h)/\partial h|_{h\to 0}$;

this holds for any L, in particular for

$|L^{y} t| = 1$, i.e. $L = t^{-1/y}$:

$m = |\,t\,|^{(d-x)/y}\,\partial f\,(\pm 1, 0)/\partial h = \text{const.}\ |\,t\,|^{\beta}$,

with critical exponent $\beta = (d-x)/y$.

Critical exponent relations from scaling

With similar arguments:

2. Susceptibility $\chi = -\partial^2 f(t, h)/\partial h^2|_{h\to 0} \sim |t|^{-\gamma}$,
   with critical exponent $\gamma = (2x-d)/y$

3. Critical isotherm $m = -\partial f(t, h)/\partial h|_{t\to 0} \sim |h|^{1/\delta}$
   with critical exponent $\delta = x/(d-x)$

4. Specific heat $(h=0)$ $C_V = -\partial^2 f(t, 0)/\partial t^2 \sim |t|^{-\alpha}$
   with critical exponent $\alpha = 2-d/y$

5. Coherence length $\xi \sim |t|^{-\nu}$
   with critical exponent $\nu = 1/y$

6. Correlation function $G \sim 1/r^{d-2+\eta}$
   with critical exponent $\eta = 2+d-2x$

which can in principle be resolved to write all
critical exponents as functions of two variables x and y.

universality classes

The critical exponents depend on only two paramters x and y.
Can these take any value?

No, because they can be shown to depend only on two other geometrical entities:

1. the spatial dimensionality d of the system
2. the dimensionality n of the order parameter

Example: Magnetization M:
$n = 1$: Ising model $s_z = \pm 1$ in $d = 1, 2, 3$ dimensions
$n = 2$: xy-model with planar spin M_{xy} moving in x-y plane
$n = 3$: Heisenberg model with 3-vector $\boldsymbol{M}$.

As d and n are discrete numbers, there is a countable number of universality classes (d, n), and within each class the critical behaviour in continuous phase transitions is identical.

Values of the critical exponents for (d, n)

with $\varepsilon = 4 - d$:

$$\gamma = 1 + \frac{n+2}{2(n+8)}\epsilon + \cdots, \tag{7.1}$$

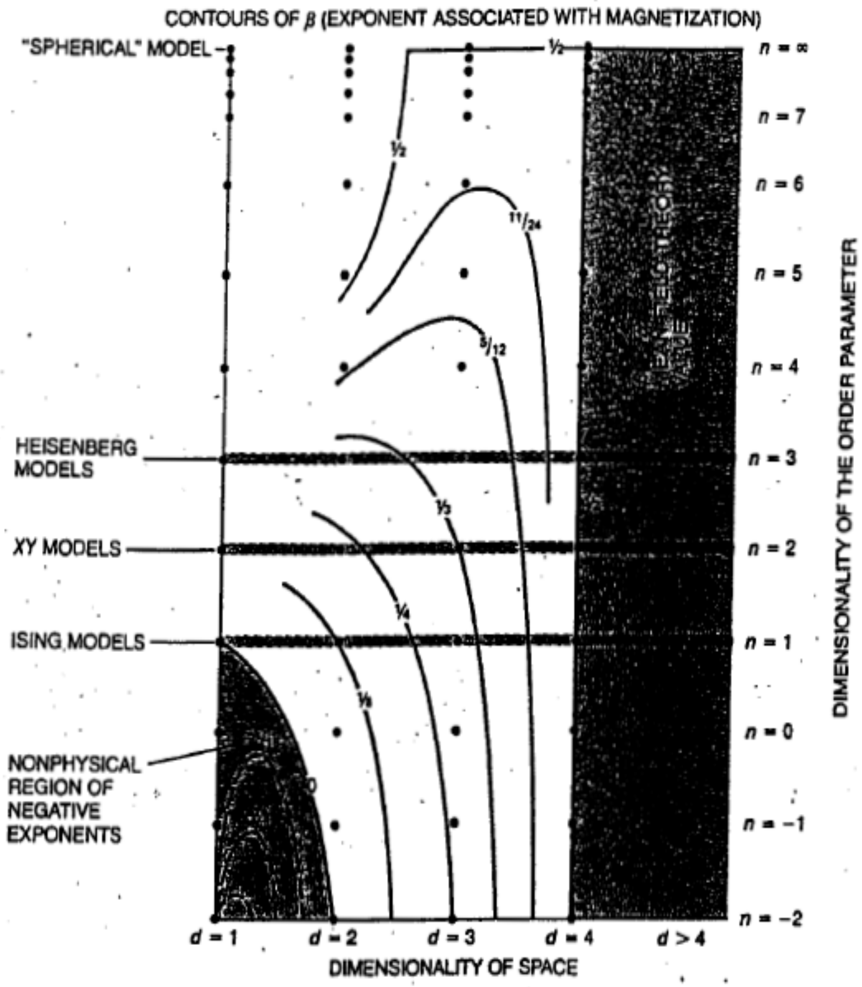
$$\beta = \tfrac{1}{2} - \frac{3}{2(n+8)}\epsilon + \cdots, \tag{7.2}$$

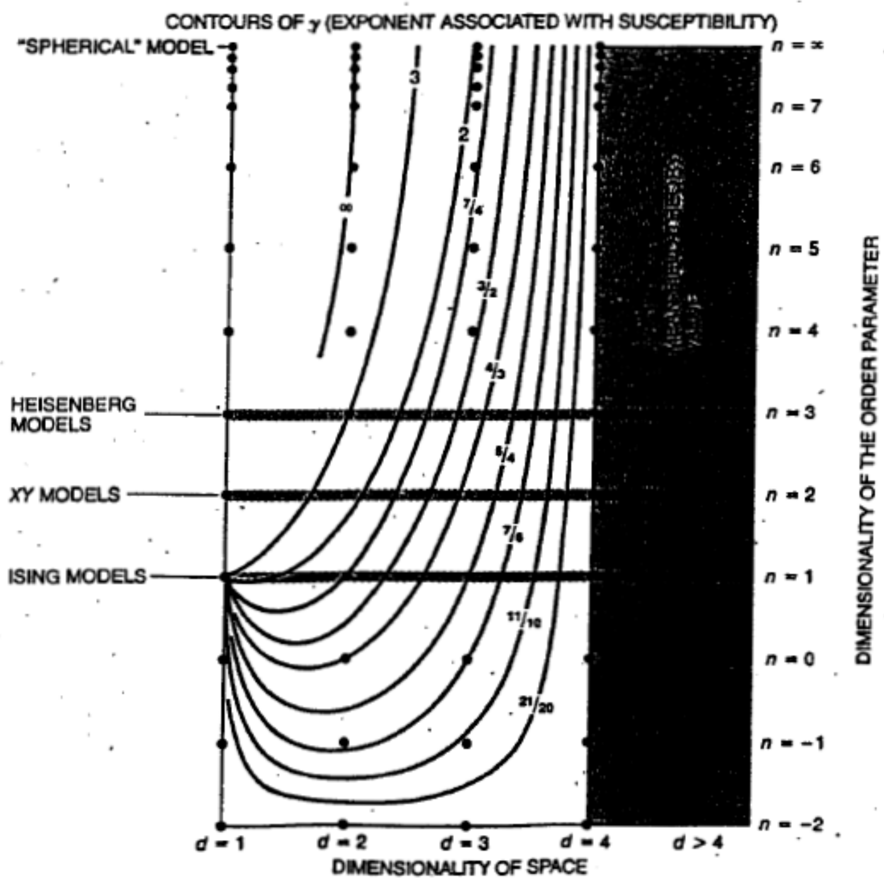
$$\alpha = \frac{4-n}{2(n+8)}\epsilon + \frac{(n+2)^2(n+28)}{4(n+8)^3}\epsilon^2 + \cdots, \tag{7.3}$$

$$\eta = \frac{n+2}{2(n+8)^2}\epsilon^2 + \cdots, \qquad \delta = 3 + \epsilon + \cdots, \tag{7.4}$$

The higher the dimension, the less the system is disturbed by fluctuations.
(example: Domino in various dimensions)
For $d = 4$, we are back at the mean field results.

Critical exponents β and γ





VARIATION OF CRITICAL EXPONENTS with the dimensionality of space (d) and of the order parameter (n) suggests that physical systems in different universality classes should have different critical properties. The exponents can be calculated as continuous functions of d and n, but only systems with an integral number of dimensions are physically possible. In a space with four or more dimensions all the critical exponents take on the values predicted by mean-field theories. The graphs were prepared by Michael E. Fisher of Cornell University.

various universality classes

Table 3.1. Universality classes

Universality class	Symmetry of order parameter	α	β	γ	δ	ν	η	Physical examples
2-d Ising	2-component scalar	0 (log)	1/8	7/4	15	1	1/4	some adsorbed mono e.g. H on Fe
3-d Ising	2-component scalar	0.10	0.33	1.24	4.8	0.63	0.04	phase separation, flu order-disorder e.g. β
3-d X-Y	2-dimensional vector	0.01	0.34	1.30	4.8	0.66	0.04	superfluids, supercon
3-d Heisenberg	3-dimensional vector	−0.12	0.36	1.39	4.8	0.71	0.04	isotropic magnets
mean-field		0 (dis.)	1/2	1	3	1/2	0	
2-d Potts, q=3	q-component scalar	1/3	1/9	13/9	14	5/6	4/15	some adsorbed mono
q=4		2/3	1/12	7/6	15	2/3	1/4	e.g. Kr on graphite

Literature

J.M. Yeomans: Statistical mechanics of phase transitions
Oxford 1992, 144 S, ca. 60 €
readable and compact

P.M. Chaikin, T.C. Lubensky: Principles of condensed matter physics
Cambridge 1995, 684 S, ca. 50 €
concise, almost exclusively on phase transitions

I.D. Lawrie: A unified grand tour of theoretical physics
Bristol 1990, 371 S, ca. 50 €
really grand tour with many analogies

P. Davies: The New Physics
Cambridge 1989, 500 S, ca. 50 €
in-bed reading

other sources will be given 'on the ride'