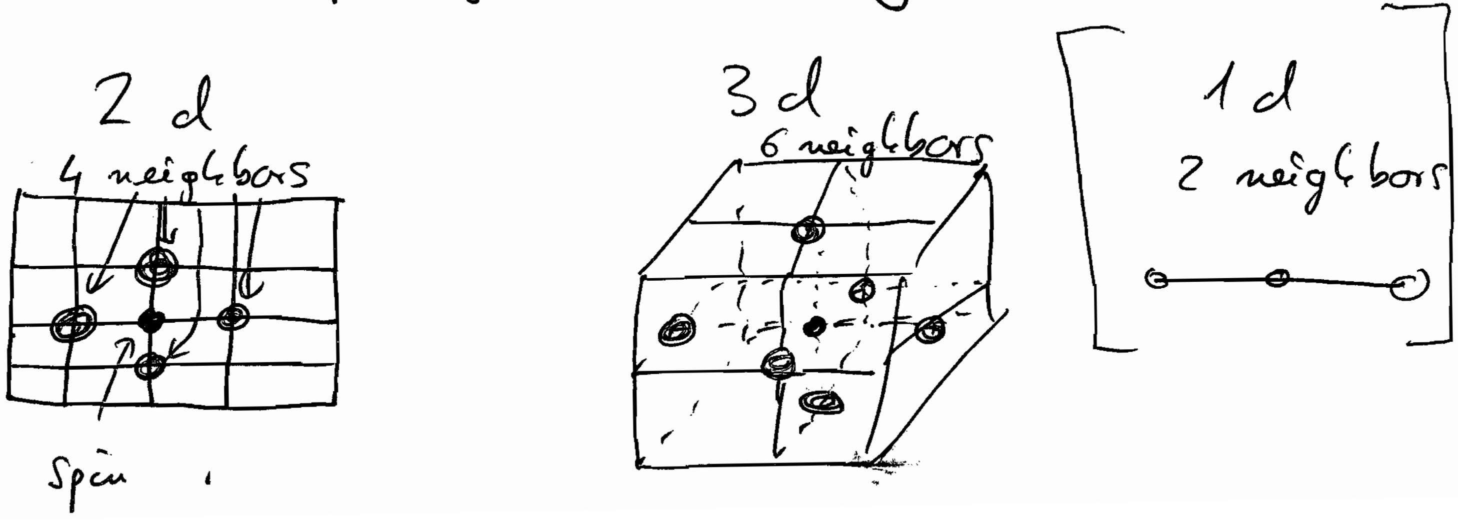
1.2 The Ising model

The Ising model is a simple model for a correlated spin system. As was already suggested in the preceding chapter, ferromagnetism is qualitatively described by a system of spins with an aligning nearest-neighbour interaction: the electron spins are in a first approx. localised on a crystal lattice. We assume a 'cubic' lattice in d dimensions,

$$H = -J \sum_{\langle ij \rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j \qquad (1.5)$$

sum over nearest neighbors

with coupling constant J.



The spins $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ with Pauli
↑ Pauli matrices
matrices $\sigma^x, \sigma^y, \sigma^z$ describe the alignment of spins pointing in any direction in $\mathbb{R}^3$ (quantum Heisenberg model). Replacing $\vec{\sigma}$ by vectors $\vec{S}$ with $\vec{S}^2 = 1$ defines the classical Heisenberg model.

This model is only soluble in $d=1$ $\Rightarrow$ further simplification: spins only have two possible orientations: spin up/down $S_i = +1/-1$.

<u>Ising model</u> (1926, proposed 1920 by Lenz to his PhD-student Ising)

$$\boxed{H = -J \sum_{\langle ij \rangle} S_i S_j \quad \text{with } S_i = \pm 1} \tag{1.6}$$

Remark:

(i) The Ising model provides, in general, a good qualitative description of spontaneous magnetisation. However, quantitatively it fails, e.g. 3d

$$\beta = 0.3265(3) \qquad \beta = 0.368(3)$$
$$\text{Ising} \qquad \text{Heisenberg} \tag{1.9}$$

Phys. Rep. Pelissetto, Vicari '02

Note also that the Heisenberg model is still a simplified one, due to anisotropies (lattice structure) and dipolar interactions.

(ii) In 2d the Ising model shows spontaneous magetisation, while the Heisenberg model does not.

Coleman-Mermin-Wagner-Hohenberg theorem

Both, the quantitative (i) and qualitative (ii) failure, are due to symmetry aspects: While the Heisenberg model has a continuous $O(3)$ symmetry, the Ising model has a discrete $O(1) \simeq \mathbb{Z}_2$ symmetry.

[Indeed, we shall see later that for 2^{nd} order phase transitions (div. correlation length ξ) the symmetries completely govern the critical physics.]

Partition function:

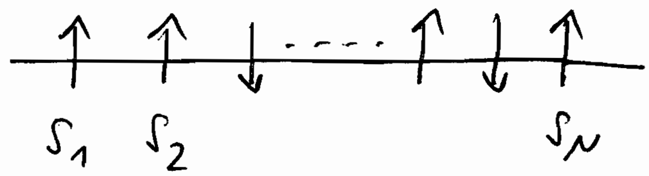
$$Z = \sum_{\{S_i\}} e^{J/k_B T \sum_{\langle ij \rangle} S_i S_j} \qquad (1.10)$$

with temperature T and Boltzmann's constant k_B.

In eq. (1.10), $\sum_{\{S_i\}}$ stands for (N = # of lattice sites)

$$\sum_{\{S_i\}} = \sum_{S_1 = \pm 1} \cdots \sum_{S_N = \pm 1}$$

Example I: one-dim Ising model



with Hamiltonian

$$H = -J \sum_{i=1}^{N-1} S_i S_{i+1} \qquad (1.11)$$

and partition function , $\boxed{\beta = \frac{1}{k_B T}}$ ← do not confuse with crit. exp. β

$$Z = \sum_{\{S_i\}} e^{-\beta H} = \sum_{\{S_i\}} e^{+\underbrace{\beta J}_{K} \sum_{i=1}^{N-1} S_i S_{i+1}} \qquad (1.12)$$

$$= \sum_{\{S_i\}} \prod_{i=1}^{N-1} e^{K S_i S_{i+1}}$$

Eq. (1.12) can be rewritten as, $\boxed{K = \beta J}$

$$Z = (\cosh K)^{N-1} \prod_{i=1}^{N-1} (1 + S_i S_{i+1} \tanh K) \qquad (1.13)$$

where we have used that ($S_i = \pm 1$)

$$e^{K S_i S_{i+1}} = \cosh K + S_i S_{i+1} \sinh K, \qquad (1.14)$$

Let us solve the product in (1.13) within a high temperature expansion, $T \to \infty$: $\beta \to 0$. This implies that $\tanh \beta J \to 0$. Hence we expand (1.13) in powers of $\tanh K$:

$$\prod_{i=1}^{N-1} \left(1 + S_i S_{i+1} \tanh K\right) = 1 + \tanh K \sum_i S_i S_{i+1} + \left(\tanh K\right)^2 \sum_{i \neq j} S_i S_{i+1} S_j S_{j+1} + \dots \qquad (1.15)$$

In front of the product displayed in (1.15) we have the sum $\sum_{\{S_i\}}$ over all possible configuration of spins. We have two cases:

(a) a given spin $S_{\hat{i}}$ only appears once in a term: $\sum_{\{S_i\}} (\dots) = 0$ as $S_{\hat{i}} = \pm 1$

(b) a given spin $S_{\hat{i}}$ appears twice / not at all
all spins appear twice / not at all.
But S_1 only appears once / not at all.

We conclude that, $K = \beta J$,

$$\boxed{\begin{aligned} Z &= \Big[\sum_{\{S_i\}}\Big] (\cosh K)^{N-1} \\ &= 2^N (\cosh K)^{N-1} \end{aligned}} \tag{1.16}$$

The partition function Z gives access to all thermodyn. functions. This allows us to determine a phase transition with Z.

However, in a finite system there is no phase transition: (a) no 1^{st} order defined by a latent heat (discontinuity)

(b) no 2^{nd} order defined by a diverging correlation length.

Finite systems do neither exhibit discont. nor infinite correlation length (by definition).

In turn, in the thermodyn. limit, $N \to \infty$, such singularities (a) or (b) may occur.

Appropriate quantity: free energy per spin $\hat{F}$:

$$\hat{F} = \lim_{N \to \infty} \frac{1}{N} F = \lim_{N \to \infty} \Big(- \frac{1}{N\beta} \ln \overset{e^{-\beta F}}{\overset{\shortparallel}{Z}} \Big) \tag{1.17}$$

$$\boxed{= - 1/\beta \ \ln 2 \cosh \beta J}$$

As $\hat{F}$ is an analytic fct. of T, the Ising model has <u>no</u> phase transition in one dimension.

General statement without proof:

'No one-dimensional system can display a phase-transition without long-range interactions'

Peierls

Correlation function:

$$\langle S_n S_m \rangle = \frac{1}{Z} \sum_{\{S_i\}} S_n S_m \, e^{-\beta H}$$

$$= \frac{1}{Z} (\cosh K)^{N-1} \sum_{\{S_i\}} S_n S_m \prod_{i=1} (1 + S_i S_{i+1} \tanh K \tag{1.18}$$

Similarily to the argument done for Z, we now have to look for the term where S_i and S_j appear once, but all spins in between twice. This leads to

$$\langle S_i S_j \rangle = \frac{1}{Z} (\cosh K)^{N-1} 2^N (\tanh K)^{|i-j|}$$

$$= (\tanh K)^{|i-j|} \tag{1.19}$$

$$\Rightarrow \boxed{\langle S_{ij} \rangle = e^{-|i-j| \ln \tanh \beta \cdot J}}$$

Eq. (1.19) entails that the correlation fct. decreases exponentially.

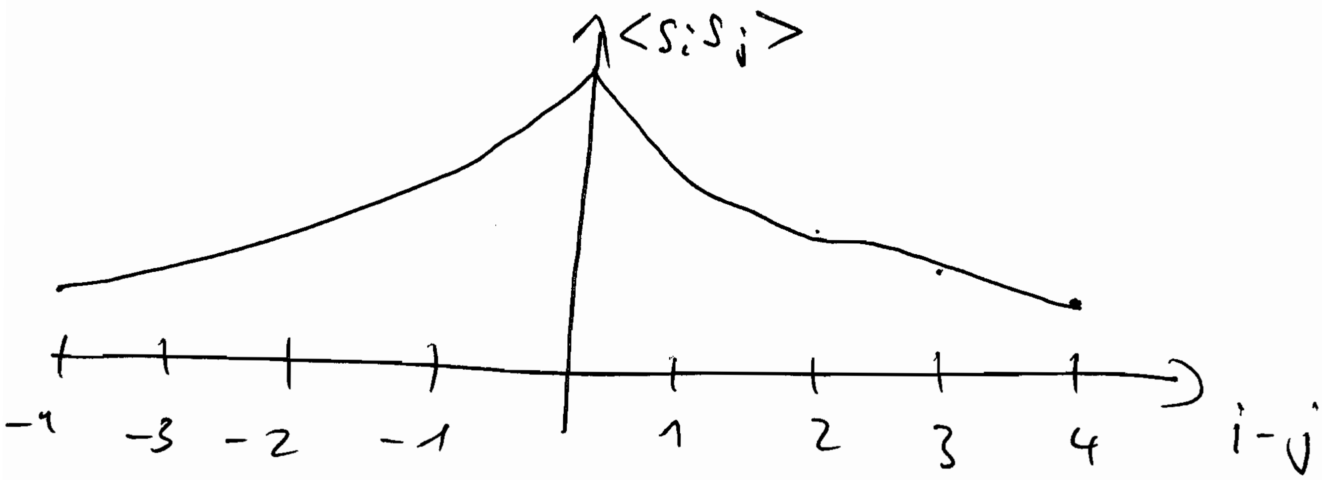
We write

$$\langle S_i S_j \rangle = e^{-r_{ij}/\xi} \qquad (1.20)$$

with $r_{ij} = |i-j| \cdot a$ and correlation length ξ,

$$\xi = \frac{a}{|\ln \tanh \beta \cdot J|} \qquad (1.21)$$

where a is the lattice spacing.



$\Rightarrow$ no spont. magnetisation, as

$$\lim_{r_{ij}\to\infty} \langle S_i S_j \rangle = \langle S_i \rangle \langle S_j \rangle = 0$$

'cluster property'

Example II : two-dim Ising model

Exact solution for Z : Onsager 1944

some details later

Transition temperature

$$\sinh 2K_c = \sinh 2J/kT_c = 1$$

$$\Rightarrow \quad T_c = \frac{2J}{k \ln(1+\sqrt{2})} \simeq 2.27\, J \tag{1.22}$$

Spin expectation value : $H = -J \sum_i s_i s_{i+1} - \boxed{\mu B \sum_i s_i}$

$$M_0 = \lim_{B \to 0_+} \lim_{N \to \infty} \frac{1}{N} \left[\sum_i \langle s_i \rangle \right]$$

arranging the spont. sym. breaking

$$= \left[1 - (\sinh 2J/kT)^{-4} \right]^{1/8} \tag{1.23}$$

Remark: The order of the limits in eq. (1.23) is <u>important</u>. First performing $B \to 0_+$ leads to $\sum_i \langle s_i \rangle = 0$!

We conclude that for $T < T_c$,

$$M \sim (T_c - T)^{1/8} \Rightarrow \beta = 1/8 \qquad (1.24)$$

We will use later this and similar exact results as benchmark tests for approximate methods.