1.3 Mean-Field approximation & critical exponents

The mean field approximation is based on the assumption that the effect of a single spin can be computed as a perturbation in the background (mean field) of the other spins.

In our case and in the presence of a background magnetic field this entails

$$H = \overbrace{-J \sum_{\langle ij \rangle} S_i S_j}^{H_0} - \mu B \sum_i S_i \quad (1.25)$$

and in mean field approx for the energy E_i of s_i is

$$E_i = J S_i \underbrace{\sum \langle s_{j_i} \rangle}_{M} - \mu B S_i \quad (1.26)$$

nearest neighbours of s_i → j_i

Now, the expectation value $\langle S_i \rangle = M$ is then given by

$$\langle s_i \rangle = \frac{\sum_{S_i = \pm 1} S_i \, e^{-\beta E_i}}{\sum_{S_i = \pm 1} e^{-\beta E_i}} \quad (1.27)$$

with $$E_i(S_i = \pm 1) = \pm\Big(-J \underbrace{\sum_{j_i} \langle S_{j_i} \rangle}_{\left[\sum_{j_i}\right] \mathcal{M}} - \mu B\Big) \qquad (1.28)$$

$$\left[\sum_{j_i}\right] = q$$

This leads to

$$\boxed{\langle S_i \rangle = \tanh \frac{q J \mathcal{M} + \mu B}{kT}} \qquad (1.29)$$

with q = # nearest neighbours, $q = 2$ in $d = 1$
$= 4$ in $d = 2$
$= 6$ in $d = 3$
$\vdots \qquad \vdots$

Since $\langle S_i \rangle = \mathcal{M}$ (constant mean field)

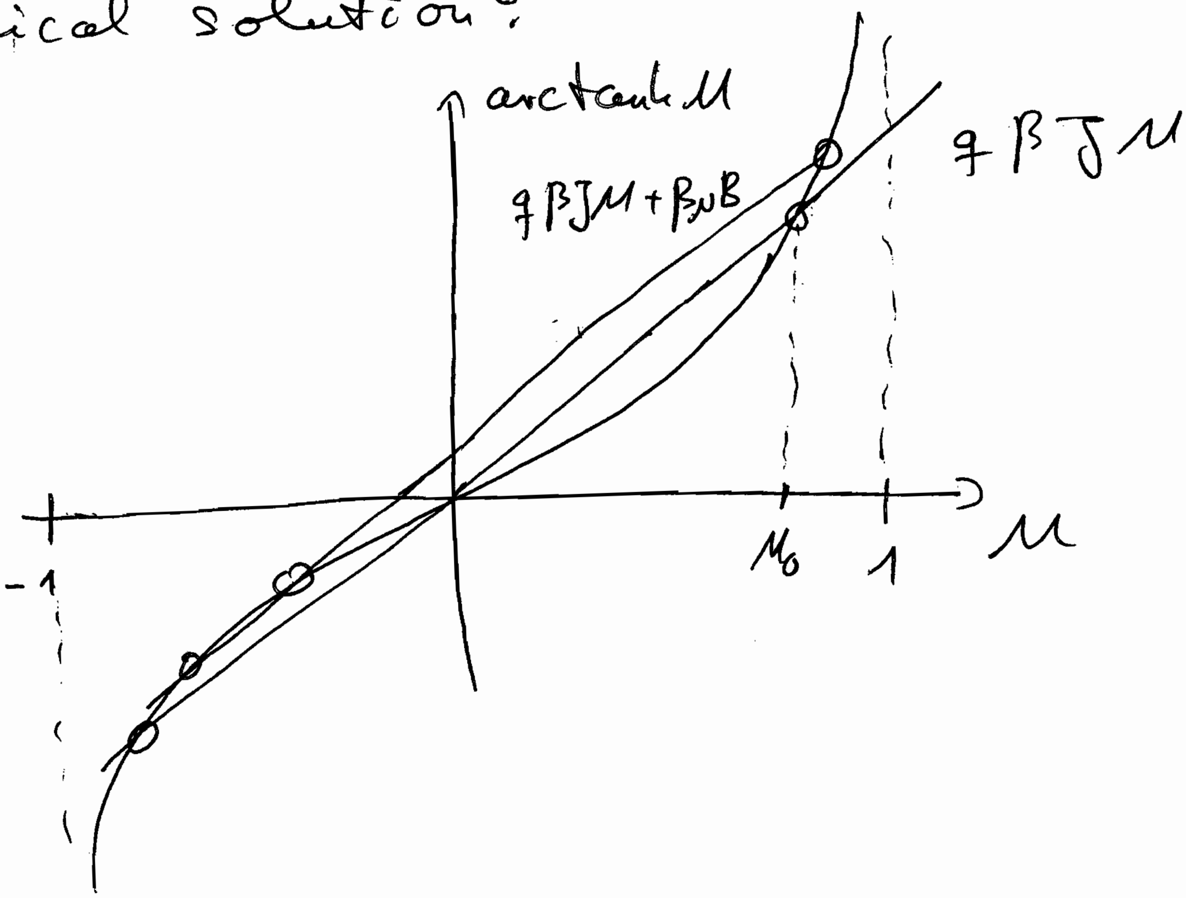
we get
$$\mathcal{M} = \tanh \frac{q J \mathcal{M} + \mu B}{kT} \qquad (1.30)$$

or

$$\boxed{\underset{\displaystyle \frac{1}{2} \ln \frac{1+\mathcal{M}}{1-\mathcal{M}}}{\underset{\shortparallel}{\operatorname{arctanh} \mathcal{M}}} = \beta\left[q J \mathcal{M} + \mu B\right]} \qquad (1.31)$$

Eq. (1.31) can now be used to study the ferromagnetic phase transition.

Graphical solution:



We consider $B > 0$: the physically stable solution then is that for $m > 0$. In the limit $B \to 0_+$ and $q\beta J > 0$ the magnetisation does <u>not</u> vanish, it tends towards $m = m_0$.
For $q\beta J < 1$, and tends towars $m = 0$.
In other words,

$$T_c = qJ/k_B: \quad \begin{array}{ll} T > T_c: & m = 0 \\ T < T_c: & m = m_0 \neq 0 \end{array} \qquad (1.32)$$

So far this gave us only numerical access to the physics of ferromagnetism. In the vicinity of the phase transition, $\mathcal{M}$ is small and we can expand all quantities about $\mathcal{M} = 0_+$:

$$\operatorname{arctanh} \mathcal{M} = \mathcal{M} + \frac{1}{3}\mathcal{M}^3 + \mathcal{O}(\mathcal{M}^5) \qquad (1.33)$$

Then, the mean field eq. reads, $(B = 0_+)$

$$\left(\mathcal{M} + \frac{1}{3}\mathcal{M}^3\right)\frac{T}{T_c} = \mathcal{M}, \qquad (1.34)$$

with $T_c = qJ/k_B$, see eq. (1.32). We introduce the reduced temp.

$$t = \frac{T - T_c}{T_c}, \qquad (1.35)$$

and rewrite (1.34) as

$$\left(\mathcal{M} + \frac{1}{3}\mathcal{M}^3\right)(1 + t) = \mathcal{M}, \qquad (1.36)$$

with the solution

$$\boxed{\mathcal{M}_0 = \sqrt{-3t}.} \qquad (1.37)$$

We conclude that near T_{c-}, the spontaneous magnetisation behaves like

$$\mathcal{M}_0 \sim (T_c - T)^{1/2} \qquad (1.38)$$

with $\boxed{\beta = 1/2}$

In 3d, this has to be compared with $\beta = 0.327..$, see eq. (1.9), page 8, in 2d $\beta = 1/8$ p. 16 Onsager

The temp. slope slope of $\mathcal{M}$ tends to ∞ for $T \to T_{c-}$: $T\partial_T \mathcal{M} \sim -\frac{1}{2}\frac{1}{(T_c-T)^{1/2}}$.

An interesting quantity is the magnetic susceptibility, $\hat{\mathcal{M}} = \mu N \mathcal{M}$,

crit. exponent

$$\chi = \left.\frac{\partial \hat{\mathcal{M}}}{\partial B}\right|_{B=0} \sim |T-T_c|^{-\boxed{\gamma}} \qquad (1.39)$$

For the B-dep. of $\mathcal{M}$ it suffices to treat the terms linear in $\mathcal{M}$ in eq. (1.31).

This amounts to, $T > T_c$

$$\mathcal{M} = (1+t)\,\mathcal{M} - \frac{\nu B}{k_B T_c} \tag{1.40}$$

and hence

$$\boxed{\mathcal{M} = \frac{\nu B}{k_B\,|T-T_c|}} \qquad \boxed{T>T_c : \mathcal{M}_0 = 0}$$

The susceptibility χ then reads

$$\chi = \frac{\nu^2 N}{k_B \boxed{(T-T_c)}} > 0 \tag{1.41}$$

For $T < T_c$ we have to take into account the spontaneous magnetisation, $\mathcal{M}_0^2 = -3t$, with

$$\mathcal{M} = \mathcal{M}_0 + \varepsilon, \tag{1.42}$$

and hence

$$\cancel{\mathcal{M}_0 + \varepsilon} = (1+t)\left[\mathcal{M}_0 + \varepsilon + \frac{1}{3}(\mathcal{M}_0+\varepsilon)^3\right] - \frac{\nu B}{k T_c} \tag{1.43}$$

$$= (1+t)(\mathcal{M}_0+\varepsilon) + \frac{1}{3}(\mathcal{M}_0+\varepsilon)^3 - \frac{\nu B}{k_B T_c} + \dots,$$

where $\dots$ stands for higher orders in the reduced tempo t.

Keeping only linear order in ε (which must be of order B) we arrive at

$$\varepsilon\left(-\mu_0^2 - t\right) = -\frac{\mu B}{k_B T_C} \Rightarrow \varepsilon = -\frac{\mu B}{2k_B(T-T_c)}, \quad (1.44)$$

where we have again dropped higher orders in t. The susceptibility follows from eq. (1.44) as

$$\chi = \frac{N\mu^2}{2k_B(T_c - T)} \sim \frac{1}{T_c - T} > 0 \quad (1.45)$$

Note that the exponents above and below T_c are the same, but the slope below T_c is half as big as above. We note in passing that the critical exponents computed so far are

$$\underset{T<T_c}{\beta = 1/2}, \quad \underset{\text{for } T \gtrless T_c\,!}{\gamma = 1} \quad (1.46)$$

A further critical exponent is derived from the isotherm at $T = T_c$: we resort to the approximation eq. (1.43) of eq. (1.31) at T_c: $t=0$

$$\mathcal{M} = \mathcal{M} + \frac{1}{3}\mathcal{M}^3 - \frac{\nu B}{k_B T_c} \Rightarrow \boxed{B = \frac{k_B T_c}{3\nu}\mathcal{M}^3,} \tag{1.47}$$

B has a cubic dependence on $\mathcal{M}$,

$$B \sim \mathcal{M}^\delta \text{ with } \delta = 3. \tag{1.48}$$

This describes the critical isotherm.

The last critical exponent we derive directly from eq. (1.31), is the behaviour of the specific heat, $C = \frac{\partial E}{\partial T}\big|_{B=0}$. The internal energy E is

$$E = \langle H \rangle = -\frac{1}{2} q J N \mathcal{M}_0^2 \qquad T < T_c$$
$$= 0 \qquad T > T_c \tag{1.49}$$

Close to the phase transition, we have $\mathcal{M}_0^2 = -3t$, eq. (1.37).

That entails
$$E = -\tfrac{1}{2} q J N \frac{3(T_c - T)}{T_c}$$

$$\boxed{qJ/T_c = k_B}_{\text{eq. (1.32)}} \rightarrow \quad = \tfrac{3}{2} k_B N (T - T_c) \,, \qquad (1.50)$$

and the specific heat follows as

$$C = \frac{\partial E}{\partial T} = \tfrac{3}{2} k_B N \sim |T - T_c|^{-\alpha} \qquad (1.51)$$

with critical exponent $\alpha = 0$. We summarize our mean-field critical exponents so far,

	observable	MF
order parameter	$M \sim \lvert t \rvert^{\beta}$	$\beta = 1/2$
susceptibility	$\frac{\partial M}{\partial B} \sim \lvert t \rvert^{-\gamma}$	$\gamma = 1$
crit. isotherm	$B \sim M^{\delta}$	$\delta = 3$
specific heat	$C \sim \lvert t \rvert^{-\alpha}$	$\alpha = 0$

(1.52)

Remarks

(i) The mean-field (MF) approximation introduced here gives predictions for the critical physics that do not depend on the dimension of the problem:

- complete failure in one dimension: mean-field predicts phase transition ↯
- working qualitatively in 2d, better in 3d, 4d fits MF-results

$T_{c\,MF} = 4J/k_B$

$T_{c\,exact} = 2.27J/k_B$

(ii) Quantum fluctuations are more violent in lower dimensions (more IR-sings.); the more neighbouring spins one has, the less influence can the quantum flucs. have.

$\boxed{\text{suppressing flucs} \rightarrow \text{favouring phase trans.}}$