1.4 Correlation functions

In eq. (1.19) we have given the expectation value of two spins, $\langle s_i s_j \rangle$ in the 1d Ising model. There, we have no spont. symmetry breaking and $\langle s_i \rangle = 0$. In general, $\langle s_i \rangle \neq 0$ and for large distances we have

$$\boxed{\lim_{|i-j|\to\infty} \langle s_i s_j \rangle = \langle s_i \rangle \langle s_j \rangle \qquad (1.53)}$$

cluster property

At infinite distance the two spins are uncorrelated. Eq. (1.53) is one of the basic axioms of <u>local</u> QFT (Euclidean: Osterwalder-Schrader).

In turn,

$$G_{ij} := \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \qquad (1.54)$$

measures the correlation of two spins.

G_{ij} is the connected 2-point correlation fct / Green fct or propagator. Its behaviour at large distances, $r_{ij} = |i-j|a \to \infty$ defines the correlation length ξ, see eq. (1.20), p. 15

$$\lim_{r_{ij}\to\infty} G_{ij} \sim e^{-r_{ij}/\xi} \qquad (1.55)$$

Similarily to the propagator G_{ij}, which measures the correlation of two spins, does

$$G_{ijkl} \overset{\langle s_i\rangle = 0}{=} \langle s_i s_j s_k s_l \rangle - \sum_{\text{permut } \sigma} \langle s_{\sigma(i)} s_{\sigma(j)} \rangle \langle s_{\sigma(k)} s_{\sigma(l)} \rangle$$

measure the non-Gaußianity of the system, i.e. for $G_{ijkl} = 0$ we have a Gaußian or free theory.

All these correlation functions derive from related generating functions.

The first such gen. fct. is the partition function in a site-dependent field B_i (source)

$$Z[B_i] = \sum_{\{S_k\}} e^{-\beta H_0 + \beta\mu \sum_k B_k S_k} \qquad (1.55)$$

with $H_0 = -J \sum_{\langle ij \rangle} S_i S_j$, see eq. (1.25).

Then, the expectation value of a spin is given by

$$\langle S_i \rangle = \frac{1}{Z} \frac{1}{\beta\mu} \frac{\partial Z}{\partial B_i} = \frac{1}{\beta\mu} \frac{\partial \ln Z}{\partial B_i} \qquad (1.56)$$

From eq. (1.56) we infer that

$$G_{ij} = \frac{1}{(\beta\mu)^2} \frac{\partial^2 \ln Z}{\partial B_i \partial B_j} = \overbrace{\frac{1}{(\beta\mu)^2} \frac{1}{Z} \frac{\partial^2 Z}{\partial B_i \partial B_j}}^{\langle S_i S_j \rangle} - \overbrace{\frac{1}{\beta\mu} \frac{1}{Z} \frac{\partial Z}{\partial B_i}}^{\langle S_i \rangle} \cdot \underbrace{\frac{1}{\beta\mu} \frac{1}{Z} \frac{\partial Z}{\partial B_j}}_{\langle S_j \rangle} \qquad (1.57)$$

In other words,

$$W[B_i] = \ln Z[B_i] \qquad (1.58)$$

generates connected correlation fcts. (Schwinger fct).

W is (−) the free energy

$$\boxed{F = -\frac{1}{\beta} W} \qquad (1.59)$$

The Gibbs free energy (thermodynamic potential) is defined as the Legendre transform of F,

$$\Gamma = F + \hat{\mathcal{M}} \cdot B = \hat{\mathcal{M}} \cdot B - \tfrac{1}{\beta} W[B] \qquad (1.60)$$

For non-uniform magnetic fields, eq. (1.60) generalises to

$$\boxed{\Gamma[\mathcal{M}_i] = \mathcal{M}_i \cdot B_i - W[B_i]} \qquad (1.61)$$

β absorbed

[strictly speaking: $\Gamma = \sup_{B_i} \{ \mathcal{M}_i \cdot B_i - W[B_i] \}$]

Eq. (1.61) implies

$$\boxed{B_i = \frac{\partial \Gamma}{\partial \mathcal{M}_i} \quad \text{and} \quad \mathcal{M}_i = \frac{\partial W}{\partial B_i}} \qquad (1.62)$$

It follows straightforwardly that

$$\sum_j \frac{\partial^2 \Gamma}{\partial \mathcal{M}_i \partial \mathcal{M}_j} \underbrace{\frac{\partial^2 W}{\partial B_j \partial B_k}}_{G_{jk}} = \delta_{ik} \qquad (1.63)$$

or $\quad G_{ij} = \left(\frac{1}{\partial^2 \Gamma / \partial \mathcal{M}^2} \right)_{ij}$.

We prove eq. (1.63) with (sum convention)

$$\frac{\partial^2 \Gamma}{\partial \mu_i \mu_j} \frac{\partial^2 W}{\partial B_j \partial B_k} = \frac{\partial}{\partial \mu_j} \underbrace{\frac{\partial \Gamma}{\partial \mu_j}}_{\parallel} \frac{\partial}{\partial B_j} \underbrace{\frac{\partial W}{\partial B_k}}_{\parallel}$$
$$= \frac{\partial}{\partial \mu_j} B_j \frac{\partial}{\partial B_j} \mu_k \tag{1.64}$$
$$= \frac{\partial}{\partial \mu_j} \mu_k = \delta_{jk}$$

Strictly speaking, eq. (1.64) only applies for $G_{ij} > 0$, that is $\lambda_i G_{ij} \lambda_j > 0$:

$$\lambda_i G_{ij} \lambda_j = \lambda_i \left[\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \right] \lambda_j$$
$$= \lambda_i \left[\langle (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle) \rangle \right] \lambda_j$$
$$= \langle S_\lambda S_\lambda \rangle \geq 0 \tag{1.65}$$

with $S_\lambda = \sum_i \lambda_i (s_i - \langle s_i \rangle)$.

Remark: Eq. (1.65) only proves that G is positive semi-definite.

In the last section we have introduced the critical exponent γ derived from the susceptibility, χ. This observable was related to the B-field dependence of the magnetisation.

The <u>response</u> of a single spin S_i to a variation of the magnetic field at the site j is given by

$$\frac{\partial\langle S_i\rangle}{\partial B_j} = \frac{1}{\beta\mu}\frac{\partial \ln Z}{\partial B_j} = \beta\mu\, G_{ij} \qquad (1.66)$$

The magnetisation $\hat{\mathcal{M}}$ related to $\langle S_i\rangle$ via

$$\hat{\mathcal{M}} = \mu\sum_i \langle S_i\rangle = \mu\langle \mathcal{S}\rangle$$

with

$$\mathcal{S} = \sum_i S_i\text{, the total spin} \qquad (1.67)$$

Hence we have

$$\frac{\partial\hat{\mathcal{M}}}{\partial B_j} = \mu\sum_i\frac{\partial\langle S_i\rangle}{\partial B_j} = \beta\mu^2\sum_i G_{ij} \qquad (1.68)$$

In the thermodyn. limit, $N \to \infty$, or for periodic boundary conditions, and for uniform B, G is translation invariant, $G_{ij} = G_{\hat{i}-\hat{j}}$. This entails
$\leftarrow$ lattice vectors belonging to i & j.

$$\frac{\partial \hat{\mu}}{\partial B} = \sum_j \frac{\partial \hat{\mu}}{\partial B_j} \frac{\partial B_j}{\partial B} \underset{\substack{\uparrow \\ B_j = B \;\; \forall j}}{=} \beta \mu^2 \sum_{i,j} G_{ij} \qquad (1.69)$$

We recall the definition of G_{ij}, eq. (1.54), and $\mathcal{S}$, eq. (1.60), and get

$$\boxed{\chi = \beta \mu^2 \sum_{i,j} G_{ij} = \beta \mu^2 \left(\langle \mathcal{S}^2 \rangle - \langle \mathcal{S} \rangle^2 \right)} \qquad (1.70)$$

Fluctuation-Response Theorem

Remarks:

(i) The fluctuation-response theorem, eq. (1.70) implies that the susceptibility, χ, is positive, $\chi > 0$.

(ii) For small correlation length, $\xi/L \ll 1$, where L is the system size, translation invariance implies

$$\chi \approx N \beta \mu^2 \sum_i G_{ij} \qquad (1.71)$$

The sums over the propagator G_{ij} relate to the Fourier transform $\tilde{G}(\vec{p})$ of G at vanishing momentum.

Consider the translation invariant case. Then, the Fourier transform $\tilde{G}(\vec{p})$ is defined as

$$\tilde{G}(\vec{p}) = \sum_{\hat{n}} G_{\hat{n}\hat{m}} \, e^{i \vec{p}(\vec{r}_{\hat{n}} - \vec{r}_{\hat{m}})} \qquad (1.72)$$

The susceptibility is given by

$$\chi = N \tilde{G}(\vec{p} = 0) \qquad (1.73)$$

for the translation invariant case.

The propagator $\tilde{G}$ also relates to the scattering cross-section of the spin system, see p. 32a.

scattering of neutrons (only spin interactions) at a ferromagnet (spin system). The scattering amplitude at the site i is given by

$$A_i \sim S_i e^{i(\vec{k}-\vec{k}')\vec{r}_i} \qquad (1.74)$$

where $\vec{k}, \vec{k}'$ are the incident and scattered wave vectors respectively (The phase of the scattering amplitude is normalised to 1 at $\vec{r}_i = 0$).

The cross section σ then reads

$$\sigma \sim \left\langle \left| \sum_i S_i e^{i(\vec{k}-\vec{k}')\vec{r}_i} \right|^2 \right\rangle = \sum_{i,j} e^{i(\vec{k}-\vec{k}')(\vec{r}_i-\vec{r}_j)}$$

$$\sim N \sum_i \left(G_{ij} + \langle S_i \rangle \langle S_j \rangle \right) e^{i(\vec{k}-\vec{k}')(\vec{r}_i-\vec{r}_j)}$$

$$\sim N \tilde{G}(\vec{k}-\vec{k}') + 2\pi N M^2 \delta_p(a(\vec{k}-\vec{k}'))$$

periodic in 2π

(1.75)

We have already seen that the propagator relates to the susceptibility. Hence we expect G to be described in the vicinity of T_c by critical exponents that are related to $\gamma, \dots$. We write ($q \ll 1/a$)

anomalous dimension

$$\tilde{G}(p) = \frac{1}{(p^2)^{1-\eta/2}} f(\xi \cdot p) \qquad (1.76)$$

monomial prefactor

dim. less fct.

We have already introduced ξ as the <u>correlation length</u>, see eq. (1.21), p. 15. For $T \to T_c$, it diverges (for a 2[nd] order phase transition) as

$$\xi \sim |T - T_c|^{-\nu} \sim |t|^{-\nu} \qquad (1.77)$$

Eq. (1.76) is not a unique splitting, we fix it by demanding

$$f(x \to \infty) = \text{const.} < \infty \quad (>0) \qquad (1.78)$$

Eq. (1.78) entails that for $T \to T_c$: $\xi \to \infty$, the propagator tends towards

$$\tilde{G}(p \to 0) \sim \frac{1}{(p^2)^{1-\eta/2}} \qquad (1.79)$$

In turn, for $p \to 0$ and $T \neq T_c$, the correlation length is finite (in the present system) and the propagator has to go to a constant, (mass gap)

$$\tilde{G}(p \to 0) \to \text{const} \qquad (1.80)$$

This entails that

$$f(x \to 0) \sim x^{2-\eta} \qquad (1.81)$$

Inserting eq. (1.81) in the propagator, we arrive at

$$\tilde{G}(0) \sim \xi^{2-\eta} = |T-T_c|^{-\nu(2-\eta)}$$

Now we recall that $\tilde{G}(0) \sim \chi \sim |T-T_c|^{-\gamma}$, thus

$$\boxed{\gamma = \nu(2-\eta)} \qquad (1.82)$$

scaling law

To see that ξ is indeed the correlation length, we go back to position space:

$$G(\vec{r}) = \int \frac{d^{\overset{\text{dimension}}{d}} p}{(2\pi)^d}\, e^{-i\vec{p}\,\vec{r}}\, \frac{f(\overbrace{\xi \cdot p}^{\hat{p} \,\leftarrow\, \text{dim. less}})}{(p^2)^{1-\eta/2}}$$

$$= \frac{1}{\xi^{d+\eta-2}} \underbrace{\int \frac{d^d \hat{p}}{(2\pi)^d}\, \frac{f(\hat{p})}{(\hat{p}^2)^{1-\eta/2}}\, e^{-i\vec{\hat{p}}\,\vec{r}/\xi}} \qquad (1.83a)$$

$$= \frac{1}{\xi^{d-2+\eta}}\, h(r/\xi) = \frac{1}{r^{d-2+\eta}}\, g(r/\xi)$$

with $g(r/\xi) = h(r/\xi)\cdot\left(r/\xi\right)^{d-2+\eta}$. (1.83b)

The definition in eq. (1.83a) implies that $h(r/\xi \to \infty) \sim e^{-r/\xi}$, hence we can identify ξ with the correlation length.

Remark: Eq. (1.83a) only implies (under weak assumptions) that $h(r/\xi \to \infty) = e^{-cr/\xi}$ with $c>0$. For $c \neq 1$ we define $\xi/c \to \xi$ as correlation length.

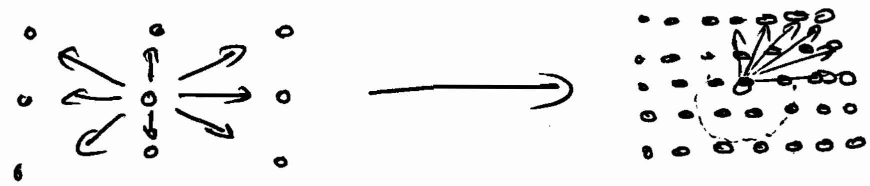
Remarks:

(i) At a second order phase transition the correlation length diverges, and the system is not determined by its microscopic interaction any more, but by its symmetry aspects & general properties: (a) symmetry of the (local) theory
(b) dimensionality

This is called <u>universality</u>.

(ii) Universality does <u>not</u> govern all aspects of the phase transition, i.e. T_c does depend on the microscopic details of the theory.

(iii) The 2d Ising model becomes isotropic in the limit $T \to T_c$. Heuristically this comes about, as the correlation length diverges so there is a finer and finer lattice grid that covers 'physically relevant' volumes:



see also Le Bellac, p. 28. Hence, at $T = T_c$ the lattice spacing drops out of physics.

A ratio of e.g. propagators reads

$$\frac{G(r_1)}{G(r_2)} = \varphi\left(r_1/r_2\,,\ r_1/a\right) \xrightarrow{T \to T_c} \varphi\left(r_1/r_2\right) \tag{1.84}$$

In other words,

$$G(sr) = \varphi(s) \cdot G(r) \tag{1.85}$$

and $\quad G(s_1 s_2 r) = \varphi(s_1)\,\varphi(s_2)\,G(r) = \varphi(s_1 s_2)\,G(r)$

Eq. (1.85) describes the behaviour of G, or other correlation fcts. under dilatations.

It implies that

$$\varphi(s) = s^{\lambda}$$

with $\lambda = d-2+\eta$, see eq. (1.76), p. 36 .