3 Renormalisation group

In the last chapter we have learned about the importance of fluctuations for the correct treatment of critical phenomena. The standard method for the treatment of quantum fluctuations, perturbation theory, fails, as it is set-up for computing small corrections to the free (Gaußian) theory, whereas critical phenomena happen in a regime of the theory which is -in general- not close to the Gaußian regime. Here we have to device methods that work away from the Gaußian regime, e.g.

(i) systematically integrating out dofs, also using the crucial scaling properties in the vicinity of a 2^{nd} order phase trans.

(ii) Expansion about the critical theory

3.1 Block-spin transformations & coarse graining

Generally, the quantum theory at hand is set-up by specifying a microscopic (classical) Hamiltonian or action as the input of the generating functional Z.
We have already introduced in chapter 2.1, p. 52,53 the concept of coarse-graining / blocking on the lattice in the context of the continuum limit. This concept goes back to Kadanoff ('66) and aims at a solution of the theory by successively integration-out degrees of freedom.
It was later applied and extended by Wilson ('71) to statistical theories / QFT's in the continuum.

Successively integrating out dofs (on the lattice) amounts to, e.g., integrating out momentum shells with (lattice) momenta

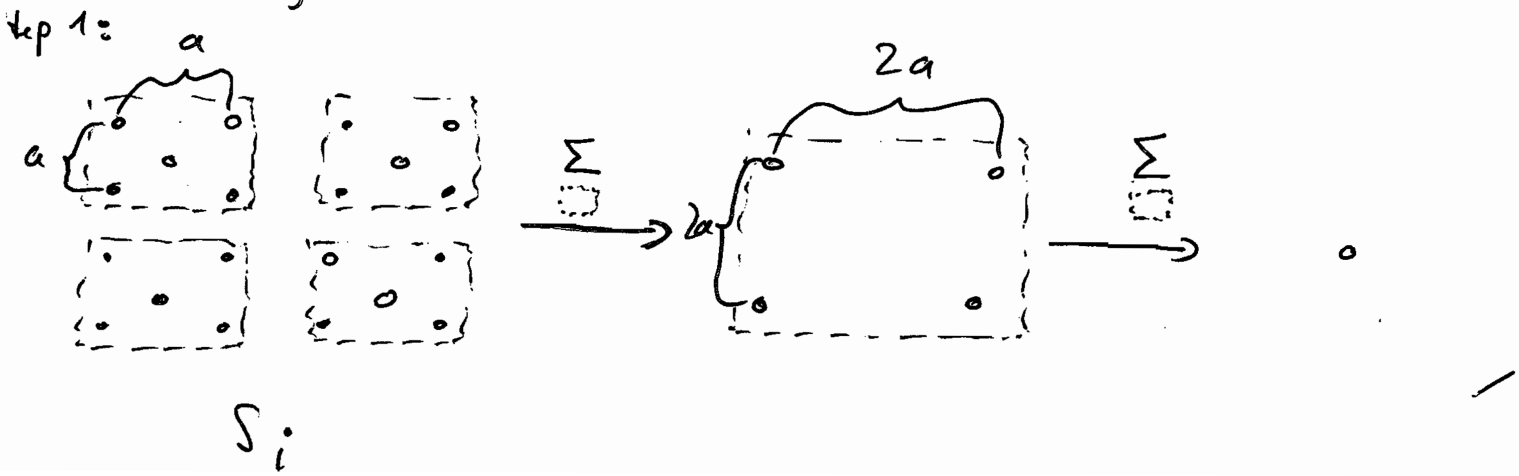
$$p \in [\pi/sa, \pi/a] \qquad (3.1)$$

with $s > 1$ and (lattice) distance a.

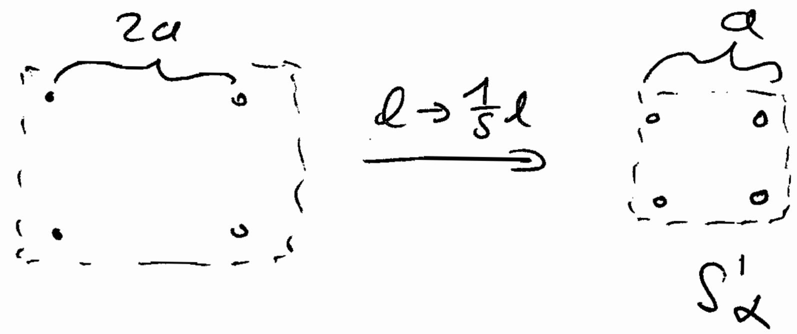
In turn we loose the resolution, as we are not sensitive any more to wave length

$$\lambda \in [a, sa] \qquad (3.2)$$

On the lattice this is implemented by summing over elementary <u>blocks</u> of the lattice, $s=2$:



Step 2: rescaling



In summary, the original theory and the blocked one live on the same lattice with lattice distance a (leave aside a finite Volume).

In analogy to eq. (2.27), p. 53 we write

$$s'_\alpha = f_\alpha(s_i)_{\text{non-linear!}} \qquad (3.2)$$

Note that in the case of a spin system with $s_i = \pm 1$ a simple averaging is not sufficient as the resulting spin variable s'_α also is required to obey $s'_\alpha = \pm 1$.

In any case, the blocking eq. (3.2) can be implemented with a δ-function: the probability for a given configuration s'_α is given by

$$e^{-H'[S'_\alpha]} = \sum_{\{S_i\}} \prod_\alpha \delta[S'_\alpha - f_\alpha(S_i)]\, e^{-H[S_i]} \qquad (3.3)$$

where H' is defined by the logarithm of the rhs. Now we use that a given configuration S_i defines $S'_\alpha = f_\alpha(S_i)$. Hence

$$\sum_{\{S'_\alpha\}} \prod_\alpha \delta[S'_\alpha - f_\alpha(S_i)] = 1 \qquad (3.4)$$

and we conclude that

$$Z = \sum_{\{S'_\alpha\}} e^{-H'[S'_\alpha]} = \sum_{\{S_i\}} e^{-H[S_i]} \qquad (3.5)$$

- General setting & Wilsonian effective action:

Let us now substitute the δ-fct. in eq. (3.3) by a more general quadratic smearing

$$\det{}^{1/2} A \cdot e^{-1/2\,(S'-f(s))\cdot A\cdot(S'-f(s))} \qquad (3.6)$$

with (invertable) smearing matrix A, e.g.

$$A_{\alpha\beta} = \cancel{b}\cdot\delta_{\alpha\beta} \qquad (3.7)$$

It is straightforward to see that for $A \to \infty$, the expression in eq. (3.6) tends towards the δ-fct. (for a cont. variable s'!)

$$\lim_{A \to \infty} \det^{1/2} A \; e^{-1/2\,(s'-f(s))\circ A\circ(s'-f(s))} = \prod_{\alpha} \delta[s'_\alpha - f_\alpha(s_i)] \qquad (3.8)$$

In any case, the $\sum_{\{s_i'\}}$ of eq. (3.8) is a constant. Hence, in summary we write

$$e^{-H_A[s']} \simeq \sum_{\{s_i\}} \det^{1/2} A \; e^{-\frac{1}{2}(s'-f(s))\cdot A\cdot(s'-f(s)) - H[s]} \qquad (3.9)$$

Remarks:

(i) The Hamiltonian or action H_A incorporates the quantum effects of the blocked areas. In turn, these areas cannot be resolved any more. In general, the Hamiltonian reads

$$-H_A[s] = K_1 \sum_{\langle ij\rangle} s_i s_j + K_2 \sum_{\langle\langle ij\rangle\rangle} s_i s_j + K_3 \sum_{\langle ijkl\rangle} s_i s_j s_k s_l + \ldots \qquad (3.10)$$

next-nearest   plaquettes::

ii) In other words, the blocking transformation R_ζ maps actions (theories) into each other.

A theory can be characterised by the set of couplings $\{K_1, \dots, K_n, \dots\}$ of the (effective) Hamiltonian/action. Accordingly a blocking transformation, or renormalisation group transformation, R_ζ acts as

$$R_\zeta \vec{\lambda} = \vec{\lambda}' \quad \text{with } \vec{\lambda} = \{K_1, \dots\}, \quad \vec{\lambda}' = \{K_1', \dots\} \tag{3.11}$$

In our case we have

$$\vec{\lambda} = \{K_1 = -J, 0, \dots\}$$
$$\vec{\lambda}' = \{K_1', K_2', \dots\} \tag{3.12}$$

iii) In the following we shall assume that an RG transformation R_ζ does not introduce long range fluctuations: the correlation between two spins decays exponentially (locality). <u>Ultra-locality</u> implies finite range interactions.

(iv) Iteration & Fixed-points. The R-transformation can be iterated,

$$R_s^n = \underbrace{R_s \circ R_s \cdots R_s \circ R_s}_{n\text{-times}}, \qquad (3.13)$$

wich is <u>not</u> R_{sn}! A fixed point H_* or λ_* under the blocking transformation has the property

$$R_s H_* = H_*, \quad R_s \lambda_* = \lambda_* \qquad (3.14)$$

Evidently, eq. (3.14) is satisfied for

$$H_* = R_s^\infty H, \quad \vec{\lambda}_* = R_s^\infty \vec{\lambda}, \qquad (3.15)$$

if it exists.

Gaussian (relevant) example:

Consider a fermionic lattice model with Euclidean symmetry (relativistic) and action

spinor indices

$$H[\psi,\bar\psi] = \sum_{\substack{i,j\\ \xi,\xi'}} \bar\psi_i^{\xi} \mathcal{D}_{ij}^{\xi\xi'} \psi_j^{\xi'} \qquad (3.16)$$

lattice points

with $\underline{\text{local}}$ Dirac operator $\mathcal{D}$. Locality means

$$\lim_{|x_i - x_j| \to \infty} \mathcal{D}_{ij}\, e^{|x_i - x_j|/\xi} \to 0 \qquad \text{for } \xi < \infty \qquad (3.17)$$

The blocked and smeared generating functional reads

$$e^{-H'[\eta,\bar\eta]} \simeq \int \mathcal{D}\psi \mathcal{D}\bar\psi \; e^{-H[\psi,\bar\psi] - (\bar\eta - \bar\psi_f)\circ A \circ (\eta - \psi_f)}$$

Grassman integrals on lattice sites

(3.18)

with

$$(\bar\eta - \bar\psi_f)\circ A \circ (\eta - \psi_f) = \sum_{\substack{i,j\\ \xi,\xi'}} (\bar\eta_i - \bar\psi_f)_i^{\xi} A_{ij}^{\xi\xi'} (\eta - \psi_f)_j^{\xi'}$$

and $\quad \psi_{f\,i} = \sum_j f_{ij}\, \psi_j \qquad (3.19)$

linear blocking

Massless fermions enjoy chiral symmetry,

$$\begin{aligned} \psi &\to e^{+i\alpha\gamma_5}\psi = \psi^\alpha \\ \bar\psi &\to \bar\psi\, e^{+i\alpha\gamma_5} = \bar\psi^\alpha \end{aligned} \quad \Rightarrow H[\psi^\alpha, \bar\psi^\alpha] = H[\psi, \bar\psi]$$

or $\boxed{\{\gamma_5, \mathcal{D}\} = 0} \qquad (3.20)$

How does 'chiral symmetry' react to blocking?

To that end we write infinitesimally

$$\delta_\alpha \psi_j = \underbrace{i\alpha(\gamma_5\cdot\psi)_i^\xi \frac{\partial}{\partial \psi_i^\xi}}_{\delta_\alpha}\psi_j \qquad (3.21)$$

and hence

$$e^{-H'[\eta + \delta_\alpha\eta,\, \bar\eta + \delta_\alpha\bar\eta]}$$

due to linear blocking, eq (3.19)

$$= \int \mathcal{D}\psi\, \mathcal{D}\bar\psi\; e^{-H[\psi,\bar\psi] - (\bar\eta - \bar\psi_f)(1+\alpha\gamma_5)\cdot A\circ(1+\alpha\gamma_5)(\eta-\psi)}$$

invariant

$$(3.22)$$

Remark: measure only inv. on the lattice!

It follows from eq. (3.22) in $O(\alpha)$ with

$$H'[\eta,\bar\eta] = \sum_{\substack{i,j\\ \xi,\xi'}} \bar\eta_i^{\xi}\, \hat{\mathcal{D}}_{ij}^{\xi\xi'}\, \eta_j^{\xi'} \qquad (3.23)$$

that

$$e^{-H'}\,\bar\eta\cdot\{\gamma_5,\hat{\mathcal{D}}\}\cdot\eta = \int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{-H}$$

$$\cdot\left(\left(\bar\eta-\bar\psi_t\right)\cdot\gamma_5\cdot\frac{\partial}{\partial\bar\eta} - \frac{\partial}{\partial\eta}\cdot\gamma_5\cdot\left(\eta-\psi_t\right)\right) e^{-(\bar\eta-\bar\psi_t)\cdot A\cdot(\eta-\psi_t)}$$

with $\operatorname{Tr}\gamma_5 = 0$

$$= \frac{\partial}{\partial\eta}\{\gamma_5, A^{-1}\}\frac{\partial}{\partial\bar\eta}\int\mathcal{D}\psi\mathcal{D}\bar\psi\, e^{-H-(\bar\eta-\bar\psi_t)\cdot A\cdot(\eta-\psi_t)} \qquad (3.24)$$

and finally

$$\boxed{\{\gamma_5,\hat{\mathcal{D}}\} = \hat{\mathcal{D}}\{\gamma_5, A^{-1}\}\hat{\mathcal{D}}} \qquad (3.25)$$

This is the Ginsparg-Wilson relation ('82). Its standard form is obtained with $A_{ij} \sim \delta_{ij}$.

Then in its dimensionful form it reads ($\hat{\mathcal{D}} \to \frac{1}{a}\hat{\mathcal{D}} = \bar{\mathcal{D}}$

$$\boxed{\{\gamma_5,\bar{\mathcal{D}}\} = 2a\,\bar{\mathcal{D}}\gamma_5\bar{\mathcal{D}}} \qquad (3.26)$$

show

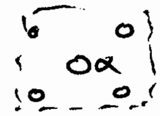
(v) linear blocking:

We have already used a linear blocking in our fermionic example, see eq. (3.19), where it was crucial for our derivation, see eq. (3.22).

In the Ising model it amounts to

$$S'_\alpha = \sum_i f_{\alpha i}\, S_i \qquad (3.27)$$

$$\text{with} \quad f_{\alpha i} = \begin{cases} \frac{\lambda(s)}{s^d} & i \text{ in } \alpha\text{-block} \\ 0 & \text{else} \end{cases}$$



This extends straightforwardly to general $f_{\alpha i}$. The factor s^{-d} divides out the rescaling of the elementary box. $\lambda(s)$ accounts for potential deviations of this <u>canonical</u> scaling.

Note that S'_α now also takes values $\neq \pm 1$!

Indeed, s'_α tends towards a continuous field ϕ_α

even though formally this only happens after infinite many $\mathcal{R}$-steps.

(vi) Correlation length after blocking:

$$\xi' = \xi / s \tag{3.28}$$

To show eq. (3.28) we compute the prop.

$$\langle s'_\alpha s'_\beta \rangle = \frac{1}{Z} \sum_{\{s'\}} s'_\alpha s'_\beta \, e^{-H'[s']}$$

eq. (3.3), p. 71 $\longrightarrow$

$$= \frac{1}{Z} \sum_{\{s'\}} \sum_{\{s\}} s'_\alpha s'_\beta \prod_\gamma \delta[s'_\gamma - f \circ s] \, e^{-H[s]}$$

$$= \frac{1}{Z} \sum_{\{s\}} (f \circ s)_\alpha (f \circ s)_\beta \, e^{-H[s]} \tag{3.29}$$

linear blocking $\rightarrow$

$$= f_{\alpha i} f_{\beta j} G_{ij} \qquad G_{ij} = \langle s_i s_j \rangle - \underbrace{\langle s_i \rangle \langle s_j \rangle}_{= 0}$$

Now we use that for $|x_\alpha - x_\beta| \to \infty$ we also have $|x_i - x_j| \to \infty$ as $f_{\alpha i}$ is only non-zero for i in the α-block, see eq. (3.27), p. 77:

$$\lim_{\underbrace{|x_i - x_j|}_{r_{ij}} \to \infty} G_{ij} \sim e^{-r_{ij}/\xi} \tag{3.30}$$

Hence

$$\lim_{r_{\alpha\beta}\to\infty} \langle S'_\alpha S'_\beta \rangle \sim f_{\alpha i} f_{\beta j}\, e^{-r_{ij}/\xi}$$

$$\sim \sum_i f_{\alpha i} \sum_j f_{\beta j}\, e^{-r_{\alpha\beta}/\underbrace{(\xi/s)}_{\xi'}}$$

$$\boxed{r_{\alpha\beta} = r_{ij}/s} \longrightarrow \quad = \frac{\lambda(s)^2}{s^d}\, e^{-r_{\alpha\beta}/\xi'} \tag{3.31}$$

with $\xi' = \xi/s$, eq. (3.28). More generally we have for $r \gg a$,

$$\boxed{G(r/s, \vec{\lambda}') \simeq \lambda^2(s)\, G(r, \vec{\lambda})} \tag{3.32}$$

The <u>linear</u> RG or blocking transformation eq. (3.27) then leads to

$$\boxed{\mathcal{R}_{s_1} \circ \mathcal{R}_{s_2} = \mathcal{R}_{s_1 \circ s_2}} \tag{3.33}$$

and hence

$$\lambda(s_1)\, \lambda(s_2) = \lambda(s_1 \cdot s_2) \tag{3.34a}$$

with solution

$$\lambda(s) = s^{d_\varphi} \tag{3.34b}$$

with anomalous dimension d_φ.

(vii) Critical surface & fixed points

Consider now a general microscopic theory, determined by its coupling vector

$$\vec{\lambda} = (\lambda_1, \ldots .) \qquad (3.35)$$

with a finite number of non-vanishing couplings $\lambda_1, \ldots, \lambda_{n_{max}}$. In the Ising model we have

$$\lambda_1 = K_1 = J/\beta, \quad \lambda_{i>1} = 0 \qquad (3.36)$$

In the 2d Ising model, the critical temperature is given by

$$T_c = 2.27/(Jk) \qquad (3.37)$$

or $K_{1c} = 1/2.27 \simeq 0.44$. In general we have

$$\vec{\lambda}_c = (\lambda_{1c}, \ldots, \lambda_{n_{max}c}, 0, \ldots) \qquad (3.38)$$

in a given theory with a second order phase transition. Applying a RG-transformation R_s (linear or non-linear (nec. in Ising model)) to it, does not change the infinite correlation length:

$$\xi = \infty \rightarrow \xi' = \xi/s = \infty \qquad (3.39)$$

Note however that $\vec{\lambda}$ might change, $\vec{\lambda}' = R_s \vec{\lambda} \neq \vec{\lambda}$!

The set of points with $\xi = \infty$ is called the <u>critical surface</u> S_∞ with

$$R_s S_\infty = S_\infty \qquad (3.40)$$

In turn, if $\vec{\lambda} \notin S_\infty$, an RG-transformation $(s > 1)$ leads away from the critical surface

$$\xi' = \xi/s < \xi \qquad (3.41)$$

(distance measured in the size of the correlation length).

As already mentioned, a fixed point is defined by

$$\boxed{\lim_{n\to\infty} R_s^n \vec{\lambda} = \vec{\lambda}_* \quad \text{with } \vec{\lambda} \in S_\infty} \qquad (3.42)$$

We have $\boxed{R_s \vec{\lambda}_* = \vec{\lambda}_*}$ (3.43)

Remarks:

(a) Each fixed point $\vec{\lambda}_*$ has a basin of attraction $\mathcal{D}(\vec{\lambda}_*)$ with

$$R_s^{\infty} \vec{\lambda} = \vec{\lambda}_* \quad \text{for } \vec{\lambda} \in \mathcal{D}(\vec{\lambda}_*) \qquad (3.46)$$

(b) There can be several fixed points.

(c) The position of the fixed points $\vec{\lambda}_*$ has no physical interpretation. They depend on the definition of R_s and on the parameterisation of the theory.

(d) The blocking procedure introduced applies to infrared fixed points ($s > 1$). More generally we can also discuss ultraviolet fixed points.