2 Landau theory

In the last chapter we have discussed the Ising model near to the phase transition. There, the correlation length ξ diverges; the theory looses its memory of the microscopic interaction, and, due to $\xi \to \infty$, it gets quasi-continuous.

Landau theory utilises the above properties (universality) in order to map the Ising model to a scalar field theory. This also simplifies the access to fluctuations.

In its simplest approximation, however, we shall recover the mean field results.

2.1 Ginzburg - Landau Hamiltonian

Let us start with a heuristic argument based on the mean-field equation eq. (1.31), p.19, in its cubic approximation, eq. (1.34),

$$\left(\mathcal{M} + \underbrace{\tfrac{1}{3}}_{>0} \mathcal{M}^3\right) T/T_c = \mathcal{M} \qquad (2.1)$$

Eq. (2.1) is the equation of motion (EoM) of $\mathcal{M} = \langle s_i \rangle$.

Let us derive in analogy a similar equation for a (constant) scalar mean field $\varphi_0 = \langle \varphi(x) \rangle$ of a scalar field theory with the same symmetry properties as the Ising Hamiltonian,

$$H_0(s_i) = H_0(-s_i) \longrightarrow H(\varphi) = H(-\varphi). \qquad (2.2)$$

Both theories share the underlying Z_2-symmetry

$$\varphi \to -\varphi \qquad (2.3)$$

relevant for the nature of the phase transition.

This task being set, we are led to

$$H(\varphi) = \frac{1}{2} r_0 \varphi^2 + \frac{1}{4!} u_0 \varphi^4 , (u_0 > 0) \quad (2.4)$$

see eq. (2.1)
related to stability

The generating function for this theory is given by

$$Z[B] = \int d\varphi \, e^{-H(\varphi) + B\varphi} \quad (2.5)$$

Now we solve eq. (2.5) in the saddle point approximation in an expansion about the minimum of the exponent,

$$H(\varphi) - B\cdot\varphi = H(\varphi_0) - B\varphi_0 + \underbrace{\left(H'(\varphi_0) - B\right)}_{0}\overbrace{(\varphi - \varphi_0)}^{\Delta\varphi}$$
$$+ \frac{1}{2} H''(\varphi_0)(\varphi - \varphi_0)^2 + \dots \quad (2.6)$$

It follows that

$$Z[B] = e^{-[H(\varphi_0) - B\varphi_0]} \int d\Delta\varphi \, e^{-\frac{1}{2} H''(\varphi_0)\Delta\varphi^2 + O(\Delta\varphi^3)} \quad (2.7)$$

Note that $\varphi_0 = \varphi_0(B)$.

We conclude that the (Helmholtz) free energy is

$$F = -\ln Z[B] = H(\varphi_0) - B\varphi_0 + \frac{1}{2}\ln H''(\varphi_0) + \cdots \qquad (2.8)$$

with $\boxed{H'(\varphi_0) = B}$

The Landau approximation is the lowest order saddle point approximation,

$$-W = F = H(\varphi_0) - B\varphi_0, \qquad (2.9)$$

and eq. (2.8) also tells us about the lowest order fluctuations: $\frac{1}{2}\ln H''(\varphi_0)$.

Now we see that the magnetisation $M = \frac{\partial W}{\partial B}$ is given by $\varphi_0 = \langle\varphi\rangle$: $W = \ln Z$ & $\varphi_0 = \varphi_0(B)$

$$\begin{aligned} M = \frac{\partial W}{\partial B} &= -\frac{\partial H}{\partial \varphi_0}\frac{\partial \varphi_0}{\partial B} + \varphi_0 + B\frac{\partial \varphi_0}{\partial B} \\ &= \left(B - \frac{\partial H}{\partial \varphi_0}\right)\frac{\partial \varphi_0}{\partial B} + \varphi_0 = \varphi_0 \end{aligned} \qquad (2.10)$$

We also easily compute the Gibbs free energy or effective action Γ,

$$\boxed{\Gamma = M\cdot B - W = \varphi_0 B + H(\varphi_0) - B\varphi_0 = H(\varphi_0) = H(M)} \qquad (2.11)$$

This amounts to

$$\Gamma(M) = \frac{1}{2} r_0 M^2 + \frac{1}{4!} u_0 M^4 \qquad (2.12)$$

with magnetic field

$$\boxed{B = \frac{\partial \Gamma}{\partial M} = r_0 M + \frac{1}{3!} u_0 M^3} \qquad (2.13)$$

wich is of the form of the mean-field equations eq. (2.1), if $r_0 \sim (T - T_c)$ close to the phase transition.

So far, we are neither sensitive to fluctuation nor to the fact, that the original theory involves N sites. Instead of a spin variable s_i, with $s_i = \pm 1$ we have introduced a scalar field $\varphi(x) \in \mathbb{R}$.

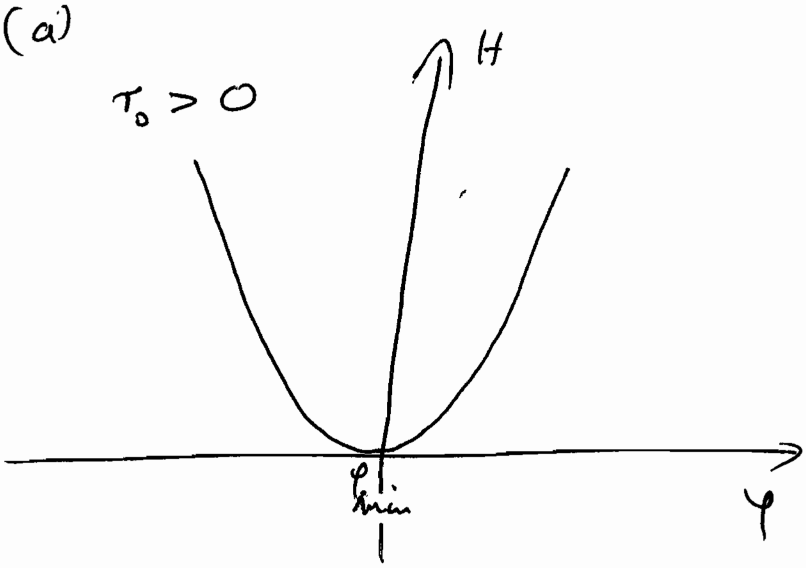
Note also that the φ^4-term effectively reduces the amplitude of the field φ.

Spontaneous symmetry breaking:

(i) Minima

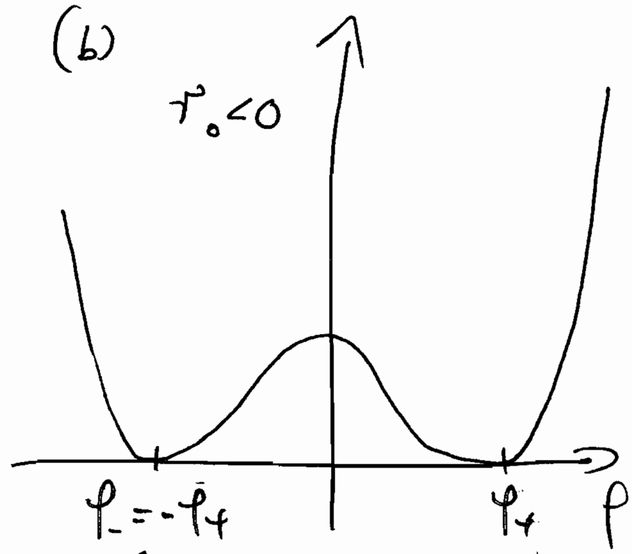
$$H(\varphi) = \frac{1}{2} r_0 \varphi^2 + \frac{1}{4!} u_0 \varphi^4$$

(a)



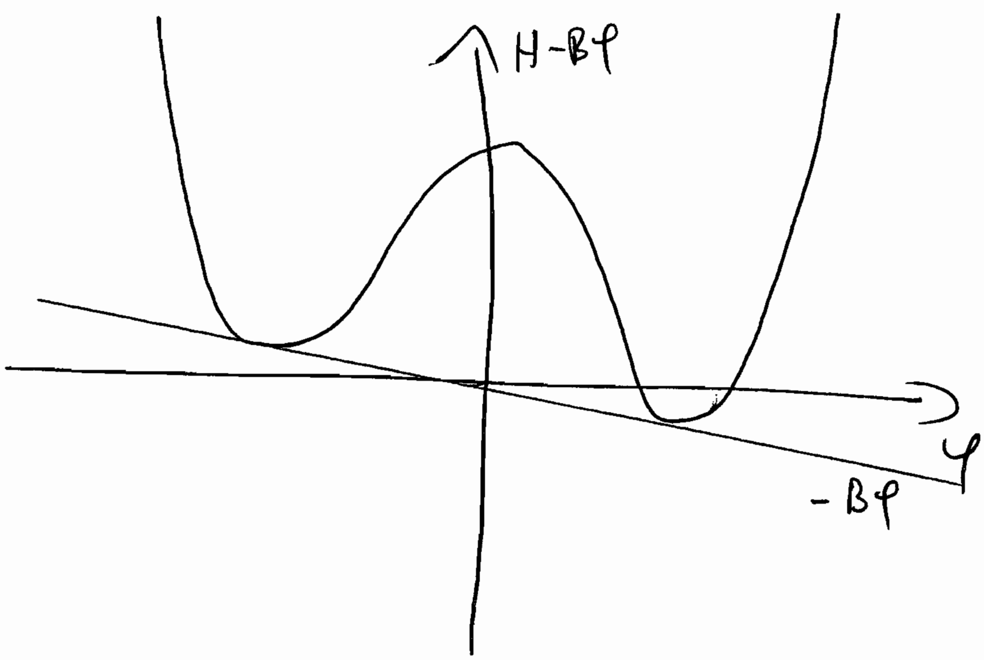
$\varphi_{min} = 0$

(b)



Z_2-symmetry $\rightarrow \varphi_{min} = \pm \varphi_+ = \pm\sqrt{-6r_0}$

(ii) $\varphi_0 = \lim_{B \to 0_+} \langle \varphi \rangle$ with weight $\exp \boxed{\{ -H(\varphi) + B \cdot \varphi \}}$



$\Rightarrow \varphi_0 = \varphi_+$ if there is no tunneling from φ_+ to φ_-

$N = 1$: evidently tunneling! see $N \to \infty$!

The nearest neighbour interaction between the spins, $\sum_i s_i s_{i+1}$ relates to the kinetic term in the field theory. To see this, we use discrete derivatives in d dimensions,

$$\nabla_\mu \varphi(\hat{x}a) = \frac{1}{a}\Big(\varphi(\hat{x}a+\hat{\mu}a) - \varphi(\hat{x}a)\Big), \qquad 2.14$$

where $\hat{x}a$ are the vectors of the lattice sites, $\hat{x} = (n_1, \ldots, n_d)$, and $\hat{\mu}$ is a unit vector in μ-direction $\mu = 1, \ldots, d$, with $\hat{\mu} = (0, \ldots, 1, \ldots, 0)$.

We write

$$-\sum_\mu \varphi(\hat{x}a)\,\varphi(\hat{x}a+\hat{\mu}a) = \frac{1}{2}\left(\nabla_\mu \varphi(\hat{x}a)\right)^2 - \frac{1}{2}\Big[d\,\varphi^2(\hat{x}a) + \sum_\mu \varphi^2((\hat{x}+\hat{\mu})a)\Big] \qquad (2.14)$$

and get the final Hamiltonian

$$\boxed{H_{GL}[\varphi(\hat{x})] = a^d \sum_{\hat{x}} \Big\{\frac{1}{2}(\nabla_\mu \varphi)^2 + \frac{1}{2} r_0(T)\,\varphi^2 + \frac{1}{4!} u_0 \varphi^4\Big\}} \qquad (2.15)$$

the Ginzburg-Landau Hamiltonian.

Remarks:

(i) The field φ carries the momentum dimension $(d-2)/2$: $[\varphi] = \frac{d-2}{2}$. With $\varphi(\hat{x}a) = a^{1-d/2}\,\hat{\varphi}(\hat{x})$ we get

$$(\nabla_\nu \varphi)^2 = a^{-d}\left(\hat{\varphi}(\hat{x}+\hat{\nu}) - \hat{\varphi}(\hat{x})\right)^2$$

$$= a^{-d}\left(-2\hat{\varphi}(\hat{x})\,\hat{\varphi}(\hat{x}+\hat{\nu}) + \hat{\varphi}^2(\hat{x}) + \sum_\nu \hat{\varphi}^2(\hat{x}+\hat{\nu})\right)$$

$$\sim s_i s_{i+j} \tag{2.16}$$

(ii) In the continuum limit, $a \to 0$, we have

$$a^d \sum_{\hat{x}} \longrightarrow \int d^d x$$

$$\nabla_\nu \varphi(\hat{x}a) = \partial_\nu \varphi(x) \tag{2.17}$$

(iii) The mass parameter $r_0 = m^2$ receives contributions from the "$s_i s_{i+1}$"-term in eq. (2.14).

Partition function:

$$Z[B(a\hat{x})] = \int \prod_{\hat{x}} d\varphi(\hat{x})\, e^{-a^d \sum_{\hat{x}} \left\{ \frac{1}{2}(\nabla_\mu \varphi)^2 + \frac{1}{2} r_0(T)\, \varphi^2 + \frac{1}{4!}\varphi^4 \right\} + \overset{a}{\boxed{a^d \sum_{\hat{x}} B(a\hat{x})\, \varphi(a\hat{x})}}} \qquad (2.18)$$

In a saddle point approximation we get similar results to the single site case:

(i) Saddle point:

$$\boxed{a^d B(\hat{x}a) = \left.\frac{\partial H}{\partial \varphi(\hat{x}a)}\right|_{\varphi(\hat{x}a) = \varphi_0(\hat{x}a)}} \qquad (2.19)$$

(ii) Free energy:

$$-W = \boxed{F = -\ln Z = H[\varphi_0(a\hat{x})] - a^d \sum_{\hat{x}} B(a\hat{x})\, \varphi_0(a\hat{x})} \qquad (2.20)$$

(iii) Magnetisation:

$$\boxed{M(a\hat{x}) = \frac{1}{a^d} \frac{\partial W}{\partial B(a\hat{x})} = \varphi_0(a\hat{x})} \qquad (2.21)$$

(iv) Gibbs free energy / Effective action:

$$\boxed{\Gamma[M(a\hat{x})] = H[M(a\hat{x})]} \qquad (2.22)$$

Spontaneous symmetry breaking revisited: $\boxed{t_0<0}$

We now consider a homogeneous field $B \to 0_+$, after taking the thermodyn. limit $N \to \infty$.

We have

$$\varphi_0 = \lim_{B\to 0_+} \lim_{N\to\infty} \frac{1}{a^d} \frac{\partial \ln Z}{\partial B(a\hat{x})}\bigg|_{B(a\hat{x})=B} \qquad (2.23)$$

saddle point approx

$$= \lim_{B\to 0_+} \lim_{N\to\infty} \frac{e^{-H[\varphi_+]+NB\varphi_+\cdot a^d}}{e^{-H[\varphi_+]+a^d NB\varphi_+}\left(1+e^{-2a^d NB\varphi_+}\right)}\varphi_+ \left[1-e^{-2a^d NB\varphi_+}\right]$$

$V = a^d \cdot N$

φ_+ contr. $\qquad \varphi_-$-contr.

$$\Rightarrow \quad \boxed{\varphi_0 = \varphi_+ \neq 0} \qquad (2.24)$$

Evidently, the other order of limits,

$$\boxed{\lim_{N\to\infty} \lim_{B\to 0} \frac{\partial \ln Z}{\partial B(a\hat{x})}\bigg|_{B(a\hat{x})=B} = 0}.$$

Indeed, this is seen in lattice simulations for, e.g. the spin expectation value $\langle S_i \rangle$ in the Ising model in $d \geq 2$ dimensions.

Continuum limit:

We have already argued that in the vicinity of a 2^{nd} order phase transition with $\xi \to \infty$ all the physical sizes of the microscopic theory become small, in particular the lattice size.

In this case we could introduce average fields that relate to sums over the field values in a box $\mathcal{B}_{b,x}$ around a given <u>continuous</u> space-time point x:

$$\varphi(x) = \frac{1}{N_b} \sum_{\hat{x}a \in B_{b,x}} \varphi(\hat{x}a) \qquad (2.25)$$

also called a coarse-graining. This only can work if

$$b/\xi << 1 \qquad (2.26)$$

Note that eq. (2.25) can be smoothed by defining

$$\varphi_f(x) = \sum_{\hat{x}} f(x-\hat{x})\,\varphi(\hat{x}a) \qquad (2.27)$$

with a smearing or (inverse) blocking fct. f.

If either $\xi \to \infty$ or $a \to 0$, we then get

$$\prod_{\hat{x}} d\varphi(a\hat{x}) \to \prod_{x} d\varphi(x) =: \mathcal{D}\varphi(x) \qquad (2.28)$$

where $\mathcal{D}\varphi(x)$ is the flat functional measure

Together with eq. (2.17) we get

$$\boxed{Z[B(x)] = \int \mathcal{D}\varphi(x)\, e^{-H_{GL}[\varphi(x)] + \int d^dx\, B(x)\varphi(x)} \qquad (2.29)}$$

with Ginzberg-Landau hamiltonian

$$\boxed{H_{GL}[\varphi(x)] = \int d^dx \left\{ \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{2}r_0(T)\varphi^2 + \frac{1}{4!}u_0\varphi^4 \right\} \qquad (2.30)}$$

The effective action / Gibbs free energy is now a functional of $M(x)$, which can be expanded about the classical Landau-Ginzburg hamiltonian

$$\Gamma[M(x)] = \int d^d x \left\{ \frac{1}{2} (\partial_\nu M)^2 + \frac{1}{2} r_0(T) M^2 + \frac{1}{4!} u_0 M^4 \right\} + \Delta\Gamma \tag{2.31}$$

where $\Delta\Gamma$ incorporates fluctuations.