2.4 Fluctuations & the Ginzburg criterion

In the derivation of the effective action $\Gamma(M)$ in the lowest order (saddle-point approx.) we have dropped the second order fluctuation term $\frac{1}{2} H''(\varphi_0)(\varphi-\varphi_0)^2$ in eq. (2.6), p. 44.

Now we go beyond this approximation in the full theory with $Z[B(x)]$ in eq. (2.29), p. 53:

$$\varphi_0(x): \quad \boxed{\left.\frac{\delta H_{GL}}{\delta\varphi(x)}\right|_{\varphi_0(x)} = B(x)} \tag{2.46}$$

and hence

$$H_{GL}[\varphi] - \underbrace{\int d^d x}_{\int_x} B(x)\varphi(x) = \overset{H_{GL}}{\overset{\parallel}{H}}[\varphi_0] + \int_x B(x)\varphi_0(x)$$

$$+ \int_x \underbrace{\left(\frac{\delta H}{\delta\varphi(x)}[\varphi_0] - B(x)\right)}_{0}\left(\varphi(x)-\varphi_0(x)\right)$$

$$+ \frac{1}{2}\int_{x,y} \frac{\delta^2 H}{\delta\varphi(x)\delta\varphi(y)}\left(\varphi(x)-\varphi_0(x)\right)\left(\varphi(y)-\varphi_0(y)\right)$$

$$+ O\left((\varphi-\varphi_0)^3\right) \tag{2.47}$$

Eq. (2.47) leads to, $\varphi_0 = \varphi_0[B]$

$$Z[B(x)] = e^{-H[\varphi_0] + \int_x B(x)\varphi_0(x)} \cdot \overbrace{\int \mathcal{D}\varphi(x)\, e^{-\frac{1}{2}\int_{x,y} \frac{\delta^2 H}{\delta\varphi^2}\cdot(\varphi-\varphi_0)^2}}^{\det^{-1/2} \frac{\delta^2 H}{\delta\varphi^2}} + O((\varphi-\varphi_0)^3) \tag{2.48}$$

and hence for the Schwinger functional

$$W[B(x)] = \ln Z = -H[\varphi_0] + \int_x B\cdot\varphi_0 - \frac{1}{2}\ln \mathrm{Tr}\, \frac{\delta^2 H}{\delta\varphi^2} + \ldots \tag{2.49}$$

In eq. (2.49) we have used that

$$\ln \det^{-1/2} \frac{\delta^2 H}{\delta\varphi^2} = -\frac{1}{2} \overbrace{\ln \det \frac{\delta^2 H}{\delta\varphi^2}}^{\ln \prod_n \lambda_n} = -\frac{1}{2} \overbrace{\mathrm{Tr} \ln \frac{\delta^2 H}{\delta\varphi^2}}^{\sum_n \ln \lambda_n} \tag{2.50}$$

Finally, the effective action reads

$$\boxed{\Gamma[M(x)] = H[M(x)] + \frac{1}{2}\mathrm{Tr}\ln \frac{\delta^2 H}{\delta\varphi^2} + \ldots} \tag{2.51}$$

see p. 61a

$$\Gamma[\mathcal{M}(x)] = \int_x B(x)\,\mathcal{M}(x) - W[B(x)]$$

$$= \int_x B\cdot\mathcal{M} + H[\varphi_0[B]] - \int_x B\cdot\varphi_0[B] + \frac{1}{2}\,\mathrm{Tr}\ln\frac{\delta^2 H}{\delta\varphi^2}[\varphi_0] + \cdots$$

$$= \int_x B\cdot\mathcal{M} + H[\mathcal{M}] - \int_x B\cdot\mathcal{M} + \frac{1}{2}\,\mathrm{Tr}\ln\frac{\delta^2 H}{\delta\varphi^2}[\mathcal{M}] + \cdots$$

$$+ \underbrace{\left(H[\varphi_0] - \int_x B\cdot\varphi_0 - H[\mathcal{M}] + \int_x B\cdot\mathcal{M}\right)}_{\boxed{= \mathcal{O}((\mathcal{M}-\varphi_0)^2) \text{ as } \frac{\delta H}{\delta\varphi_0} - B = 0}}$$

with $\mathcal{M}(x) = \varphi_0(x) + \mathcal{O}(u_0)$, $\mathcal{M}(x) = \dfrac{\delta W}{\delta B}$

The $\cdots$ stand for sub-leading corrections, i.e.

$$r_0 \to r_0 \quad \left[\frac{1}{2}\,\mathrm{Tr}\ln\frac{\delta^2 H}{\delta\varphi^2}\right]^{(2)}[\mathcal{M}=0] + \mathcal{O}(u_0^2)$$

where we have introduced

$$\frac{\delta^n F}{\delta\varphi^n} = F^{(n)}[\varphi]$$

Evaluating eq. (2.51) for constant magnetisation leads to

$$\boxed{\frac{1}{V}\Gamma(M) = H(M) + \frac{1}{2}\underbrace{\int\frac{d^d q}{(2\pi)^d}\ln\left(p^2+r_0(T)+\frac{u_0}{2}M^2\right)}_{\text{see p. 62a}} + \ldots} \quad (2.52)$$

Remarks: (i) Eq. (2.52) is not finite term by term reflecting the necessity of renormalisation

(ii) This also includes the M-independent terms. Note that they are important for the thermodyn. potential and cannot be simply dropped!

We postpone the issue of renormalisation and discuss M-derivatives of $\Gamma(M)$, indeed we have $\frac{\partial\Gamma}{\partial M} = B$, $\frac{\partial\Gamma}{\partial M^2} = \tilde{G}^{-1}(0)$.

$$\frac{1}{2}\, \mathrm{Tr}\left[\ln \frac{\delta^2 H}{\delta\varphi^2}\right](x,y)$$

Tr in momentum space

$$= \frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\left[\ln \frac{\delta^2 H}{\delta\varphi^2}\right](p,p)$$

uniform M

$$= \frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\left[\ln\left(p^2 + r_0(T) + u_0/2\, M^2\right)\right]\underbrace{(2\pi)^d\,\delta(p-p)}_{\int d^dx\, e^{i(p-p)x} = V}$$

$$= V/2 \int \frac{d^dp}{(2\pi)^d} \ln\left(p^2 + r_0(T) + u_0/2\, M^2\right)$$

It follows

$$B = r_0(T)\mu + \frac{u_0}{6}\mu^3 + \frac{1}{2} u_0 \mu \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2 + r_0(T) + \frac{1}{2}u_0\mu^2} \qquad (2.53)$$

and the susceptibility for $T > T_c$: $\mu_0 = 0$

finite

$$r := \underset{\text{is } |T-T_c|^{-\gamma}}{\chi^{-1}} = \underset{\parallel \; \frac{\partial^2 \Gamma}{\partial \mu^2}}{\left.\frac{\partial B}{\partial \mu}\right|_{B=0}} = r_0(T) + \frac{u_0}{2}\mu_0^2 + \frac{u_0}{2}\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2 + r_0(T) + \frac{1}{2}u_0\mu_0^2}$$

$$+ \mathcal{O}(\mu_0^2)\text{-terms}$$

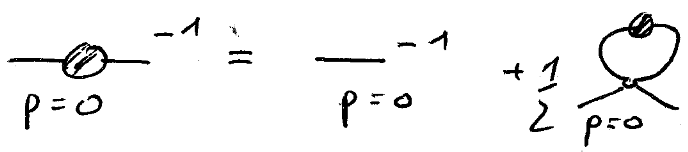
$$= r_0(T) + \frac{u_0}{2}\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2 + r_0(T)} \qquad (2.54)$$

and hence $T_c \neq T_0$! For the comp. of T_c we rewrite eq. (2.54) as

$$r = r_0(T) + \frac{u_0}{2}\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2 + r} \qquad (2.55)$$

[that is the DSE

$$\tilde{G}^{-1}(0) = \tilde{G}_0^{-1}(0) + \frac{1}{2}u_0 \int \frac{d^dp}{(2\pi)^d}\tilde{G}(p) \qquad (2.56)$$



[QFT II chapter 3.2, p.60 ff]

in the present approximation]

At the phase transition, χ diverges: $r = 0$

We have with $r_0(T_c) = \bar{r}_0 (T_c - T_0)$

$$\bar{r}_0 (T_c - T_0) = -\frac{u_0}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2}$$

$$\left[= -\frac{u_0}{2} \frac{\Omega_d}{(2\pi)^d} \int_0^{\Lambda \,\leftarrow\, \text{cut-off}} dp\, p^{d-3} \right.$$

$$\left. = -\frac{u_0 \Omega_d}{2(2\pi)^d} \frac{\Lambda^{d-2}}{d-2} \right] \qquad (2.57)$$

With $\bar{r}_0 (T - T_0) = \bar{r}_0 (T_c - T_0) + \bar{r}_0 (T - T_c)$ and eq. (2.57)

we get from eq. (2.55)

$$\boxed{r = \bar{r}_0 (T - T_c) - r \overbrace{\left[\frac{u_0}{2} \int^{\Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 (p^2 + r)} \right]}^{C(r)}} \qquad (2.58)$$

$C(r)$ is only infrared finite for $r \to 0$ for $d > 4$.

(i) $d > 4$: $C(0) < \infty$

$$\Rightarrow \quad \boxed{r = \frac{\bar{r}_0 (T - T_c)}{1 + C(0)}} \qquad (2.59)$$

Critical exponents unchanged: mean-field dominates!

(ii) $d < 4$: $(d>2)$

$$C(\tau) = u_0 \tau^{(d-4)/2} \overbrace{\left[\frac{\Omega_d}{2(2\pi)^d} \underbrace{\int_0^\infty \frac{p^{d-1}\,dp}{p^2(p^2+1)}}_{-\frac{\pi}{2}\frac{1}{\sin(\pi d/2)}} \right]}^{C} \qquad (2.60)$$

With eq. (2.60) the relation (2.58) turns into

$$\boxed{\tau = \tau_0 (T-T_c) - u_0\, C \cdot \frac{\tau}{\tau^{\varepsilon/2}}} \qquad (2.61)$$

with

$$\boxed{\varepsilon = 4 - d} \qquad (2.62)$$

Evidently, eq. (2.61) is incompatible with $\tau \sim (T-T_c)$ and the critical exponents for $d<4$ cannot be computed by Landau theory!

In turn, Landau theory is valid for

$$\boxed{u_0\, C / \tau^{\varepsilon/2} \ll 1} \qquad (2.63)$$

Ginzburg criterion

Eq. (2.63) can be written as a bound on $T-T_c$ by substituting $r=\bar{r}_0(T-T_c)$:

$$\boxed{u_0 C\left[\bar{r}_0(T-T_c)\right]^{(d-4)/2} \ll 1} \qquad (2.64)$$

Eqs. (2.63), (2.64) constrain the importance of fluctuations. This is also measured by the propagator:

$$\tilde{G}(0) = \int_{x,y} \overbrace{\left[\langle \varphi(x)\varphi(y)\rangle - \langle\varphi(x)\rangle\langle\varphi(y)\rangle\right]}^{\text{mean-square fluctuations (variance)}} \qquad (2.65)$$

$$= V\int d^dx\, G(x)$$

Normalised by $[\underbrace{\int d^dx \langle\varphi(x)\rangle}_{V^2\mu_0^2}]^2$ $\leftarrow \boxed{T<T_c}$ it yields

$$\tilde{G}(0)/V^2\mu_0^2 = \xi^2/V^2 \cdot \frac{u_0}{6\bar{r}_0(T_0-T)}$$

eq. (2.35), p. 55
" (2.42) p. 59

$$\boxed{= \frac{\xi^{2d\nu V^2}}{V^2}\cdot\frac{u_0}{6}\,\bar{r}_0^{\frac{d-4}{2}}\,(T_0-T)^{\frac{d-4}{2}} \ll 1} \qquad (2.66)$$

Finiteness of $\Gamma(\mathcal{M})$:

$$\frac{1}{V}\Gamma(\mathcal{M}) \simeq \frac{1}{2} r_0 \mathcal{M}^2 + u_0/4!\, \mathcal{M}^4$$

$$+ \frac{1}{2} \int^{\Lambda} \frac{d^d p}{(2\pi)^d} \ln\left(p^2 + r_0 + u_0/2\, \mathcal{M}^2\right)$$

$$\frac{\partial \Gamma}{\partial \mathcal{M}} \simeq r_0 \mathcal{M} + u_0/3!\, \mathcal{M}^3 + \frac{u_0 \mathcal{M}}{2} \int^{\Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + r_0 + u_0/2\, \mathcal{M}^2}$$

$r_0 = \bar{r}_0 (T - T_c) + \bar{r}_0 (T_c - T_0) = $ eq. (2.57), p. 64 $\leftarrow \boxed{d > 2}$

$$\simeq \bar{r}_0 (T - T_c) \mathcal{M} + u_0/3!\, \mathcal{M}^3 + \frac{u_0 \mathcal{M}}{2} \int \frac{d^d p}{(2\pi)^d} \left[\frac{1}{p^2 + r_0 + u_0/2\, \mathcal{M}^2} - \frac{1}{p^2}\right] + O(u_0^2)$$

$$\simeq \bar{r}_0 (T - T_c) \mathcal{M} + u_0/3!\, \mathcal{M}^3 - \frac{u_0 \mathcal{M}}{2} \int \frac{d^d p}{(2\pi)^d} \frac{r + u_0/2\, \mathcal{M}^2}{p^2 (p^2 + r + u_0/2\, \mathcal{M}^2)}$$

(i) finite for $d < 4$ (superrenormalisable)

(ii) $d = 4$: renormalisation of u_0 required!

(iii) $d > 4$: non-renormalisable