3.2 Critical exponents & scaling

We have already computed the scaling behaviour for large $r \gg a$, see eq. (3.32), (3.34b). At a fixed point we have

$$G(r, \vec{\lambda}_*) = s^{-2d_\varphi}\, G(r/s, \vec{\lambda}_*) \qquad (3.45)$$

Now we choose the dilatation factor $s = r/a$.

Then

$$G(r, \vec{\lambda}_*) = \left(r/a\right)^{-2d_\varphi}\, G(a, \vec{\lambda}_*) \qquad (3.46)$$

$$\simeq r^{-d+2-\eta} \leftarrow \text{eq. (1.76), p. 36}$$

and hence

$$\boxed{d_\varphi = \frac{1}{2}\big(\underbrace{d-2}_{\text{canonical dimension}} + \underbrace{\eta}_{\text{anomalous dimension}}\big)} \qquad (3.47)$$

We conclude that the RG-transformations have to include factors $\lambda(s)$ for allowing for fixed points.

Now we want to describe the physics near a fixed point. To that end we expand our theory (in terms of the coupling vector $\vec{\lambda}$) about the fixed point theory ($\vec{\lambda}_*$), to wit

$$\vec{\lambda} = \vec{\lambda}_* + \delta\vec{\lambda} \tag{3.48}$$

Away from the fixed point we have

$$\vec{\lambda}' = R_s \vec{\lambda} \quad \text{with} \quad \vec{\lambda}' \neq \vec{\lambda} \tag{3.49}$$

and hence

$$\boxed{\begin{aligned} \delta\lambda'_i &= \sum_j T_{ij}\,\delta\lambda_j + \mathcal{O}(\delta\lambda^2) && (3.50a) \\ \text{with} \quad T_{ij}(s) &= \left.\frac{\partial\lambda'_i}{\partial\lambda_j}\right|_{\vec{\lambda}_*} && (3.50b) \end{aligned}}$$

The group property eq. (3.33), p. 78 for the (linear) RG transformation R_s implies

$$T(s_1 \circ s_2) = T(s_1) \cdot T(s_2) \tag{3.51}$$

change the infinite correlation length:

$$\xi = \infty \rightarrow \xi' = \xi/s = \infty \qquad (3.39)$$

Note however that $\vec{\lambda}$ might change, $\vec{\lambda}' = R_s \vec{\lambda} \neq \vec{\lambda}$!

The set of points with $\xi = \infty$ is called the <u>critical surface</u> S_∞ with

$$R_s S_\infty = S_\infty \qquad (3.40)$$

In turn, if $\vec{\lambda} \notin S_\infty$, an RG-transformation ($s > 1$) leads away from the critical surface

$$\xi' = \xi/s < \xi \qquad (3.41)$$

(distance measured in the size of the correlation length).

As already mentioned, a fixed point is defined by

$$\boxed{\lim_{n \to \infty} R_s^n \vec{\lambda} = \vec{\lambda}_* \quad \text{with } \vec{\lambda} \in S_\infty} \qquad (3.42)$$

We have $\boxed{R_s \vec{\lambda}_* = \vec{\lambda}_*}$ (3.43) and hence

$$T(s) = e^{B \ln s} \tag{3.52}$$

with

$$B_{ij} = s \left.\frac{dT}{ds}\right|_{s=1} \tag{3.53}$$

B is called the stability matrix. It is frequently written in terms of the β-functions of the theory

$$\boxed{\beta_i = s \frac{\partial \lambda_i}{\partial s}} \tag{3.54}$$

which leads to

$$B_{ij}(\vec{\lambda}) = \frac{\partial \beta_i}{\partial \lambda_j}(\vec{\lambda}) \tag{3.55}$$

and $B_{ij} = B_{ij}(\vec{\lambda}_*)$. Now let $e^{(i)}$ be the eigen vectors of B with eigenvalues $b_i \in \mathbb{C}$,

$$B \cdot \vec{e}^{(i)} = b_i \cdot \vec{e}^{(i)} \tag{3.56}$$

This implies that

$$\boxed{T(s) \cdot \vec{e}^{(i)} = s^{b_i} \vec{e}^{(i)}} \tag{3.57}$$

Let us assume for the moment that the $e^{(i)}$ form an orthonormal basis of space of couplings. Note that this is non-trivial as B is not necessarily symmetric! Then

$$\boxed{\begin{aligned} \delta\vec{\lambda} &= \sum_i t_i \, \vec{e}^{(i)} \\ \text{and} \quad \delta\vec{\lambda}' &= \sum_i \underbrace{t_i \, s^{b_i}}_{t_i'} \vec{e}^{(i)} \end{aligned}} \qquad (3.58)$$

The coefficients t_i are called <u>scaling</u> fields.

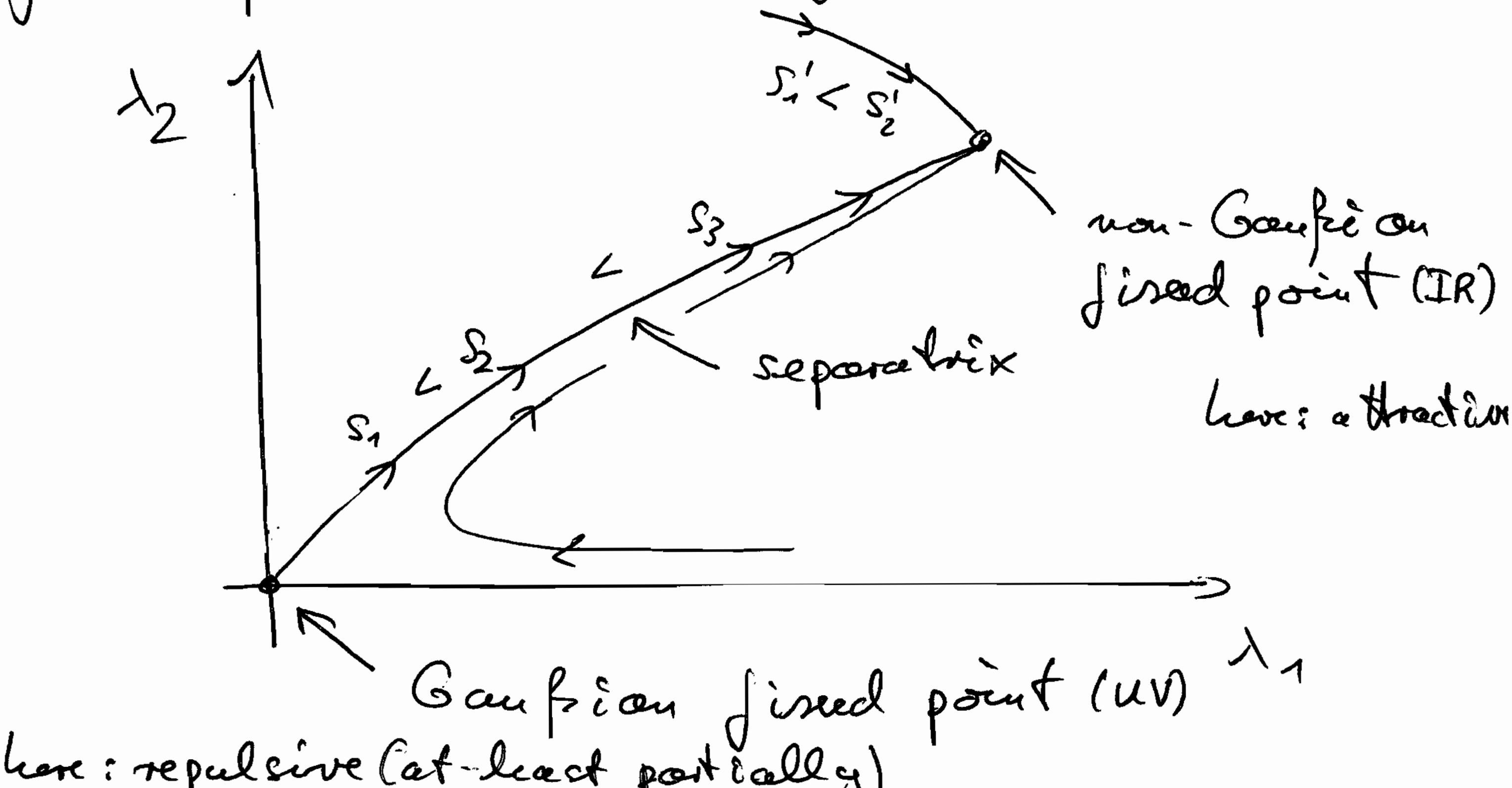
We have to distinguish three cases:

(i) $\operatorname{Re} b_i > 0$: the scaling field t_i increases under an RG-trafo, it is called a <u>relevant</u> field

(ii) $\operatorname{Re} b_i = 0$: the scaling field is constant under an RG-trafo: <u>marginal</u> field

(iii) $\operatorname{Re} b_i < 0$: the scaling field decreases under a RG-trafo: <u>irrelevant</u> field

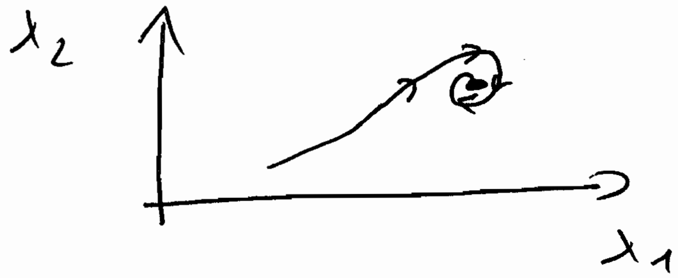
We emphasise that the above relevance counting only refers to the specific fixed point (FP) under investigation. E.g. we have



$$\vec{\lambda}(s) = \vec{\lambda}_* + \delta\vec{\lambda}(s) \tag{3.58}$$

$$= \vec{\lambda}_* = \sum_i t_i \, s^{b_i} \, \hat{e}(i)$$

where we assumed real b_i's. For complex b_i's the above picture about the non-Gaußian FP changes to



In the vicinity of a FP we can write

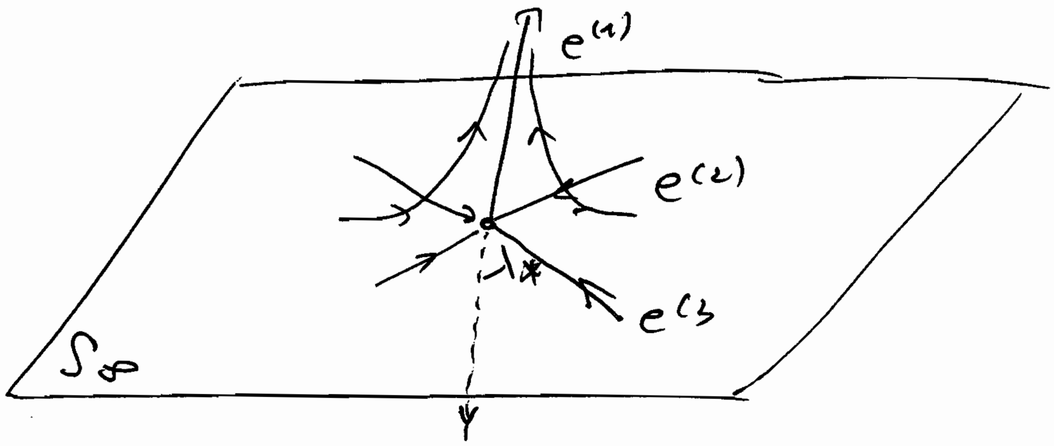
$$H = H^* + \sum_i t_i O_i \qquad (3.60)$$

where the O_i are the operators linked to the scaling fields t_i. In general, these are superpositions of the operators O_{λ_i} linked to the couplings λ_i. The O_i are called relevant/marginal/irrelevant according to the scaling of t_i.

Fixed point and critical exponents: Example

Suppose that

$$b_1 > 0, \quad 0 < b_2 \leq b_3 \leq \dots$$



Now we use that

$$G(r; t_1, t_2, \dots) = s^{-2d_\varphi} G(r/s; t_1', t_2', \dots) \tag{3.61}$$

and on S_∞: $t_1 = 0$, and $s = r$

$$G(r; t_1, t_2, \dots) = r^{-2d_\varphi} G(1; 0, t_2', \dots)$$

$$\to r^{-2d_\varphi} \underbrace{G(1, 0, 0, \dots)}_{>0}$$

We conclude that at the FP,

$$\boxed{G(r \to \infty) \sim r^{-2d_\varphi}} \tag{3.62}$$

Now let $|t_1| =: \xi^{-b_1}$ which we will identify with the correlation length. We have

$$G(r) = s^{-2d_\varphi}\, G(r/s\,;\, \pm (s/\xi)^{b_1}, s^{b_2} t_2, \dots) \qquad (3.63)$$

$$s = \xi \longrightarrow \quad = \frac{1}{\xi^{2d_\varphi}}\, G(r/\xi\,;\, \pm 1, \xi^{b_2} t_2, \dots)$$

$$\xi^{b_2} t_2 \ll 1 \longrightarrow \quad \overset{!}{=} \quad \frac{1}{r^{d_\varphi}}\, f_\pm (r/\xi)$$

above/below T_C

From eq. (1.76), p. 36 : $\tilde{G}(p) = \frac{1}{(p^2)^{1-\eta/2}} f(\xi \cdot p)$ we deduce that

$$d_\varphi = \tfrac{1}{2}(d - 2 + \eta) \qquad (3.64a)$$

and with $\xi = |t_1|^{-1/b_1}$ is indeed the correlation length with

$$\nu = 1/b_1 \qquad (3.64b)$$

The above analysis is valid in the neighbourhood of the FP $\vec{\lambda}_*$ where the linear approximation is valid.

Note also that this neighbourhood can be reached from any point $\vec{\lambda} \in \mathcal{D}(\vec{\lambda}_*)$, the basin of attraction of $\vec{\lambda}_*$. We conclude that the eqs. (3.64) display universal properties of the FP that are independent from the Hamiltonian $H(\vec{\lambda} \in \mathcal{D}(\vec{\lambda}_*))$! All these H's lead to the same η & ν for

$$t_2 \xi^{b_2} \ll 1 \Leftrightarrow \boxed{|T - T_c|^{-\nu b_2} t_2 \ll 1} \qquad (3.65)$$

Let us now extract the other critical exponents, see p. 26. To that end we switch on a constant magnetic field,

$$H \to H + B \sum_i s_i \qquad (3.66)$$

-scaling

$$\to H' + B \frac{s^d}{\lambda(s)} \sum_\alpha s'_\alpha$$

$$= H' + B\, s^{d-d_\varphi} \sum_\alpha s'_\alpha$$

Thus we define

$$B' = B\, s^{b_B} \quad \text{with} \quad \boxed{b_B = \frac{1}{2}(d+2-\eta)} \tag{3.67}$$

B is one of the couplings t_i and indeed $b_B > 0$ for $\eta < \frac{d+2}{2}$. This is true in almost all cases, and hence

$$G(r, t, B) = s^{-2d_\varphi}\, G(r/s, s^{b_1} t, s^{b_B} B), \tag{3.68}$$

where we have suppressed all irrelevant directions.

The magnetisation M reads

$$\begin{aligned} M(t, B) &= s^{-d_\varphi} M(s^{b_1} t, s^{b_B} B) \\ &= s^{-d_\varphi} M(\pm (s/\xi)^{1/\nu}, s^{b_B} B) \end{aligned} \tag{3.69}$$

For $T = T_c$: $\xi \to \infty$ and hence with $s = B^{-1/b_B}$

$$\begin{aligned} M(0, B) &= B^{d_\varphi/b_B} M(0, 1) \\ &\sim B^{d_\varphi/b_B} \end{aligned} \tag{3.70}$$

With $M \sim B^{\delta}$ it follows

$$\boxed{\delta = \frac{d+2-\eta}{d-2+\eta}} \qquad (3.71)$$

The critical exponent β follows with $T<T_c$ and $B=0$ for $s=\xi$:

$$M(t,0) = \xi^{-d\varphi} M(-1,0) \sim |t|^{-\nu d\varphi} \qquad (3.72)$$

and hence

$$\boxed{\beta = \frac{1}{2}\nu\,(d+2-\eta)} \qquad (3.73)$$

The critical exponent α, $C \sim |t|^{-\alpha}$, follows from $C = -1/V\, T^2 \frac{d^2 T\Gamma}{dT^2} \underset{\alpha>0}{\overset{T\to T_c}{\sim}} \frac{d^2\Gamma}{dt^2}$. The free energy, however, is sensitive to the constant term in Γ (or W) which we have dropped so far: We write more carefully for the blocked H',

$$e^{-(H'[S'_\alpha]+\Omega)} = \sum_{\{S_i\}} \prod_\alpha \delta\big(S'_\alpha - f(S_i)\big)\, e^{-H[S_i]} \qquad (3.74)$$

The free energy $F = -\ln Z = -W = \Gamma[\Phi_{min}]$

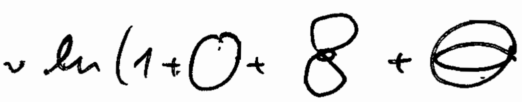
is proportional to Ω,

$$F(\vec{\lambda}) = F(\vec{\lambda}') + \Omega(\vec{\lambda}) \qquad (3.75)$$

with

$$F(\vec{\lambda}') = -\ln Z', \quad Z' = \sum_{\{s'\}} e^{-H'[s']} \qquad (3.76)$$

Remark: Diagrammatically, $\Omega(\vec{\lambda})$ comprises all vacuum diagrams,

$\Omega \sim \ln(1 +$  $+ \dots)$

Now we do our blocking steps, starting from a initial theory with $\vec{\lambda}_0 \in \mathcal{D}(\vec{\lambda}_\infty)$.

We get for

$$f(\vec{\lambda}) := F(\vec{\lambda})/L^d \qquad (3.77)$$

which divides out the volume factor, with eq. (3.75)

$s=2$

$$f(\vec{\lambda}_0) = 2^{-d} f(\underset{\overset{\shortparallel}{\vec{\lambda}'}}{\vec{\lambda}_1}) + \omega(\vec{\lambda}_0) \qquad (3.78)$$

where we have used that $L' = L/s$ and

$$f(\vec{\lambda}') = F(\vec{\lambda}')/(L/2)^d \quad \text{for } s=2 \tag{3.79}$$

For the ith RG-step we get from eq. (3.78)

$$f(\vec{\lambda}_{i-1}) = 2^{-d} f(\vec{\lambda}_i) + \omega(\vec{\lambda}_{i-1}) \tag{3.80}$$

and hence by reinserting eq. (3.80) on the rhs of eq. (3.78),

$$\boxed{f(\vec{\lambda}_0) = \sum_{n=0}^{\infty} (2^{-n})^d \cdot \omega(\vec{\lambda}_n)} \tag{3.81}$$

In the limit of continuous RG-transformations this reads

$$\boxed{f(\vec{\lambda}_0) = \int_0^{\infty} \frac{ds}{s}\, s^{-d}\, \omega(\vec{\lambda}(s))} \tag{3.82}$$

The integral in eq. (3.82) is a function of $\boxed{s^{b_1} \cdot t}$, as $\omega(\vec{\lambda}(s)) = \hat{\omega}(s^{b_1} t)$ with

$t(s) = s^{b_1} t$ and $t = t(\vec{\lambda}_0)$. With a rescaling, $\hat{s} = s|t|^{1/b_1}$ we arrive at

$$\boxed{f(\vec{\lambda}_0) = |t|^{d/b_1} \underbrace{\int_0^{\infty} \frac{d\hat{s}}{\hat{s}}\, \hat{s}^{-d}\, \hat{\omega}(\pm \hat{s})}_{A_\pm:\ \text{coefficient}}} \tag{3.83}$$

We conclude that the free energy scales like

$$f(\vec{\lambda}_0) \sim |t|^{\nu d} \qquad (3.84)$$

with $\nu = 1/b_1$. With $C \sim \frac{d^2 f}{dt^2}$ we have

$$\boxed{\alpha = 2 - \nu d} \qquad (3.85)$$

Summary of scaling laws:

$$\begin{aligned}
C \sim |t|^{\alpha} &: \quad \alpha = 2 - \nu d \\
M \sim |t|^{\beta} &: \quad \beta = \tfrac{1}{2}\nu(d-2+\eta) \\
\frac{\partial M}{\partial B} \sim |t|^{\gamma} &: \quad \gamma = \nu(2-\eta) \\
B \sim M^{\delta} &: \quad \delta = \frac{d+2-\eta}{d-2+\eta}
\end{aligned} \qquad (3.86)$$

There are only 2 independent critical exponents, e.g. ν and η, which govern the critical behaviour of the 2-point correlator, $\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle$. Evidently, the other observables in the first column in eq. (3.86) can be derived from it.

Exercise