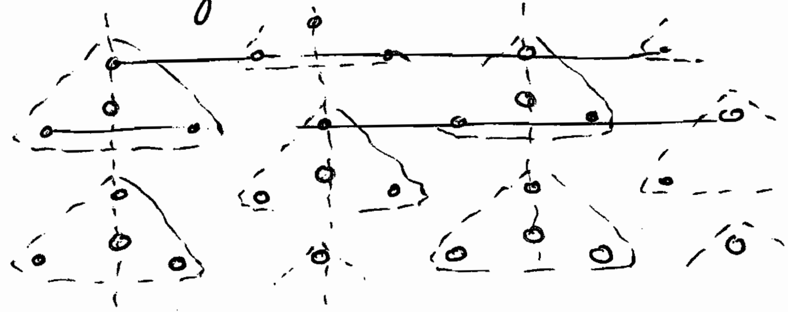
3.3 Approximation by cumulants

We consider the Ising model in two dim. on a triangular lattice



This allows to define a simple, <u>non-linear</u>, RG-transformation

$$S'_\alpha = \text{sign}\left[S_\alpha^{(1)} + S_\alpha^{(2)} + S_\alpha^{(3)}\right] = f_\alpha(S_i) \qquad (3.87)$$

Evidently, eq. (3.87) maps spin variables $S_i = \pm 1$ into spin variables $S_\alpha = \pm 1$.

The scaling factor is $\boxed{s = \sqrt{3}}$ but the map does not preserve the angles!

The new hamiltonian reads

$$e^{-\Omega - H'[S']} = \sum_{\{S\}} \prod_\alpha \delta[S'_\alpha - f_\alpha(S_i)]\, e^{-H[S]} \qquad (3.88)$$

The δ-fcts. in eq. (3.88) fix the spin configs. in each block to obey $f_\alpha(s_i) = s'_\alpha$. Hence the hamiltonian H separates naturally into

$$\underset{\text{interaction inside blocks}}{H = H_0} + \overset{\text{interaction between blocks}}{V} \tag{3.89}$$

where H_0 stands for interactions in a given block and V mediates interactions between blocks.

We use eq. (3.89) to reorder eq. (3.88) as

$$\boxed{e^{-\Omega - H'[s']} = \frac{\sum_{\{s\}} e^{-H_0} \prod_\alpha \delta[s'_\alpha - f_\alpha(s_i)] \sum_{\{s\}} e^{-H_0 - V} \prod_\alpha \delta[s'_\alpha - f_\alpha(s_i)}{\sum_{\{s\}} e^{-H_0} \prod_\alpha \delta[s'_\alpha - f_\alpha(s_i)]}} \tag{3.90}$$

The normalisation/prefactor $\sum e^{-H} \prod \delta[\;]$ is straightforward to compute: one has to sum over all configs $\{s_i\}$ that satisfy the constraint with the weight factor $\sum_{\{s^{(i)}_\alpha\}} \delta[s'_\alpha - f_\alpha(s_i)]\, e^{-H_0} = Z_0$.

The remaining term is the expectation value of the operator e^{-V} in the theory with hamiltonian H_0 (with restricted configs. due to $\prod \delta[\ \ ]$). We write

$$\langle \mathcal{O} \rangle_0 = \frac{\sum_{\{S\}} \mathcal{O}[S]\, e^{-H_0[S]} \prod_\alpha \delta[S'_\alpha - f_\alpha(S)]}{\sum_{\{S\}} e^{-H_0[S]} \prod_\alpha \delta[S'_\alpha - f_\alpha(S)]} \quad (3.91)$$

These considerations lead to

$$\boxed{\begin{aligned} & e^{-\Omega - H'[S']} = Z_0 \cdot \langle e^{-V} \rangle_0 \\ & \text{with} \quad Z_0 = \sum_{\{S\}} e^{-H_0[S]} \prod_\alpha [S'_\alpha - f_\alpha(S)] \end{aligned}} \quad (3.92)$$

Note that $\langle \mathcal{O} \rangle_0$ rather is $\langle \mathcal{O} \rangle_0 [S']$. Even though Z_0 is simple to compute, $\langle e^{-V} \rangle_0$ is not. Indeed e^{-V} couples <u>all</u> blocks (V^n mediates couplings of n neighbouring blocks). For the sake of simplicity, let us expand $\langle e^{-V} \rangle$ in powers of V.

To that end we use that

$$\ln \langle e^{-V} \rangle = \ln\left(\langle e^{-(V-\langle V \rangle)} \rangle e^{-\langle V \rangle} \right)$$

$$= -\langle V \rangle + \ln \langle 1 - (V - \langle V \rangle) + \tfrac{1}{2}(V-\langle V \rangle)^2$$

$$- \tfrac{1}{3!}(V - \langle V \rangle)^3 + \dots \rangle$$

$$= -\langle V \rangle - (V - \langle V \rangle)^2 + \mathcal{O}((V - \langle V \rangle)^3) \tag{3.100}$$

As indicated above, the comp. of the higher terms gets increasingly involved. Here, we already drop the second term, to wit

$$\langle e^{-V} \rangle_0 = e^{-\langle V \rangle_0} \tag{3.101}$$

which is, again, a mean-field approximation.

It is left to compute $\langle V \rangle_0$ & Z_0:

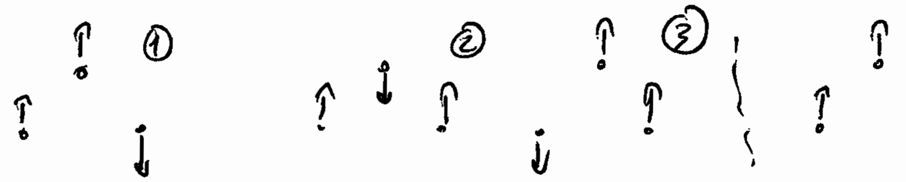
(i) Z_0 factorises as H_0 only has interactions inside blocks by definition.

$$Z_0 = \prod_\alpha Z_{Block}[S'_\alpha] \tag{3.102}$$

However, Z_{Block} is independent of S'_α as it is quadratic in the spins S_i:

$$\boxed{Z_{Block} = e^{3K} + 3e^{-\overset{=\beta J}{K}}} \qquad (3.103)$$

with



(with $S' = 1$)

Hence

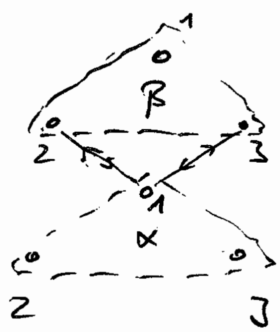
$$Z_0 = Z_{Block}^{\underset{\uparrow}{N/3}} = \left(e^{3K} + 3e^{-K}\right)^{N/3} \qquad (3.104)$$

\# of Blocks

(ii) Computation of $\langle V \rangle_0$ where

$$V_{\alpha\beta} = -K\, S_\alpha^{(1)} \left(S_\beta^{(2)} + S_\beta^{(3)} \right) \qquad (3.105)$$

between 2 blocks α, β with



As H_0 does not mediate interactions between different blocks, we have

$$\langle S_\alpha^{(1)} S_\beta^{(i)} \rangle = \langle S_\alpha^{(1)} \rangle \langle S_\beta^{(i)} \rangle \qquad (3.106)$$

and we only have to evaluate the expectation values of single spins S_i for a given $S'_{\alpha/\beta}$ with the Hamiltonian H_0. This leads to

$$\boxed{\begin{aligned} \langle S_\alpha^{(1)} \rangle_0 &= \frac{1}{Z_{Block}} \sum_{\{S_\alpha^{(i)}\}} S_\alpha^{(1)} e^{K\left[S_\alpha^{(1)} S_\alpha^{(2)} + S_\alpha^{(1)} S_\alpha^{(3)} + S_\alpha^{(2)} S_\alpha^{(3)}\right]} \\ &= \frac{1}{Z_{Block}}\left[\left(e^{3K} + e^{-K}\right)\delta_{1 S'_\alpha} - \left(e^{3K} + e^{-K}\right)\delta_{-1 S'_\alpha}\right] \end{aligned}} \tag{3.107}$$

where we have used that for $S'_\alpha = 1$:

$$\begin{array}{cccc} 1\ 2\ 3 & 1\ 2\ 3 & 1\ 2\ 3 & 1\ 2\ 3 \\ \uparrow\ \uparrow\ \uparrow & \uparrow\ \uparrow\ \downarrow & \uparrow\ \downarrow\ \uparrow & \downarrow\ \uparrow\ \uparrow \\ S_\alpha^{(1)}=1 \downarrow & \downarrow & \downarrow & \downarrow \\ +e^{3K} & +e^{-K} & +e^{-K} & -e^{-K} \end{array}$$

$$= e^{3K} + e^{-K}$$

and similarily for $S'_\alpha = -1$: $-\left(e^{3K} + e^{-K}\right)$.

In other words

$$\boxed{\langle S_\alpha^{(1)} \rangle_0 [S'_\alpha] = \frac{1}{e^{3K} + e^{-K}}\left(e^{3K} + e^{-3K}\right) S'_\alpha} \tag{3.108}$$

With eq. (3.109) and eq. (3.108) we get

$$\boxed{\begin{aligned}\langle V_{\alpha\beta}\rangle_0 &= 2K\left(\frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\right)^2 S_\alpha' S_\beta' \\ &= K' S_\alpha' S_\beta' \qquad (3.109) \\ \text{with} \quad K' &= 2K\left(\frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\right)^2\end{aligned}}$$

This leads to the final blocked Hamiltonian

$$\boxed{e^{-\Omega - H'[S']} = e^{N/3 \ln\left(e^{3K}+3e^{-K}+K'\sum_{\langle\alpha\beta\rangle} S_\alpha' S_\beta'\right)}} \qquad (3.110)$$

The Hamiltonian in eq. (3.110) allows for a simple fixed point analysis. The FP is given by

$$\boxed{K_* = 2K_*\left(\frac{e^{4K_*}+1}{e^{4K_*}+3}\right)^2} \qquad (3.111)$$

which can be solved algebraically.

We get

$$e^{K_*} = \underbrace{1 + 2\sqrt{2}}_{x}$$

and

$$K_* \simeq 0.336 \quad (\text{exact } 0.275) \qquad (3.112)$$

As we have only one coupling there is only one scaling exponent b:

$$s^b \overset{\nearrow}{=} \left.\frac{\partial K'}{\partial K}\right|_{K_*} = 2\left(\frac{x+1}{x+3}\right)^2 + \frac{32 K_* x}{(x+3)^2}\left(\frac{x+1}{x+3}\right) \qquad (3.113)$$

eq. (3.50), p. 83

With $s = \sqrt{3}$ and $s^b \simeq 1.634$ we conclude

$$\nu = 1/b = 1 \Big/ \frac{\ln 1.634}{\ln \sqrt{3}} = 1.118 \qquad (3.114)$$

(exact 1)

We summarize these results in a table

	exact (Onsager)	1st order cumulant	MF
K_*	0.275	0.336..	1/6
ν	1	1.118..	1/2