4.1 Flow equation

So far we have blocked (and smeared) the field over a box in space-time. This effectively restricts the allowed wave-vectors or momenta. Indeed, starting with continuous space-time fields, $\varphi(x)$ with $x \in \mathbb{R}^d$, blocking is represented by

$$\varphi_f(\hat{n}a) = \int d^dx \, f_a(\hat{n}a - x)\, \varphi(x).$$

where

$$f_a(x) = \begin{cases} 1/a^d & |x_\mu| \leq a/2 \\ 0 & \text{else} \end{cases} \tag{4.1}$$

This introduces a lattice with lattice-distance a and the momenta are restricted to the Brillouin zone,

$$p_\mu \in [-\pi/a, \pi/a] \tag{4.2}$$

Under the scaling

$$p \to sp \,, \quad x \to x/s \qquad \text{beware of momentum counting!} \qquad (4.6)$$

the field transforms as

$$\varphi(x) \to \varphi'(x') = \lambda(s)\,\varphi(x) = s^{d_\varphi}\varphi(x)$$

$$\Rightarrow \tilde{\varphi}(p) \to \tilde{\varphi}'(k') = \lambda(s)\, s^{-d/2}\,\tilde{\varphi}(k) = s^{d_\varphi - d}\,\tilde{\varphi}(k) \qquad (4.7)$$

The above can be summarised in 3 RG-steps:

(i) Integration over momentum shell

$$\Lambda/s \leq p \leq \Lambda \qquad (4.8a)$$

(ii) Rescaling of the unit of length

$$p \to p' = s\,p$$

$$x \to x' = x/s \qquad (4.8b)$$

(iii) Re-normalisation of the field

$$\tilde{\varphi}(k) \to \tilde{\varphi}'(k') = s^{d_\varphi - d/2}\,\tilde{\varphi}(k)$$

$$\varphi(x) \to \varphi'(x') = s^{d_\varphi}\,\varphi(x) \qquad (4.8c)$$

Further blocking, $a \to a' = sa$, reduces the momenta to

$$p'_\mu \in [-\pi/sa, \pi/sa] \qquad (4.3)$$

We see that continuous rescalings with $s \in \mathbb{R}^+$ easily introduced in momentum space,

$$\boxed{p_\cdot \to s\, p_\cdot \,, \quad s \in \mathbb{R}^+} \qquad (4.4)$$

in other words, we restrict the momentum integration to momenta smaller or bigger than a cut-off scale $\underline{\Lambda}$ (UV) or $\underline{k}$ (IR) respectively. With such a setting in mind, it is more convenient to formulate everything in momentum space. We define in a finite volume

$$\tilde{\varphi}(p) = \frac{a^d}{L^{d/2}} \sum_x e^{ipx} \varphi(x) \qquad (4.5)$$

$$\xrightarrow{a \to 0} \int d^d x\, e^{ipx} \varphi(x)$$

The new Hamiltonian is then given as

$$e^{-\Omega - H'[\varphi']} = \int \prod_{\Lambda/s \le p \le \Lambda} d\tilde{\varphi}(p)\, e^{-H[\varphi]} \Big|_{\tilde{\varphi} \to s^{d_h - d_\varphi} \tilde{\varphi}'(sp)} \qquad (4.9)$$

Within this setting, we easily re-derive the scaling law for the propagator. To that end we take $p < \Lambda/s$. The physics of these momentum modes is not changed by the integration in eq. (4.9). Hence the propagator is unchanged, we only have to perform the steps (ii) and (iii), to wit

$$\boxed{\tilde{G}(sp; \vec{\lambda}') = s^{2d_\varphi - d}\, \tilde{G}(p, \vec{\lambda})} \qquad (4.10)$$

For $x \gg 1/\Lambda$, we can approximately perform the Fourier trafo:

$$\boxed{G(x/s; \vec{\lambda}') = s^{2d_\varphi}\, G(x; \vec{\lambda}) \quad (4.11)}$$

Soft cut-off:

The integration measure in eq. (4.9) only involves momentum modes $\tilde{\varphi}(p)$ with $1/s \leq p \leq 1$. Eq. (4.9) is not well-suited for practical computations. We shall reformulate the mode integration within a more general formulation that also includes soft cut-off choices in which modes are only suppressed but not completely removed.

Quantum field theories are given/determined by a complete set of correlation functions.

Example: scalar field theory with a real field $\varphi(x)$ in d dim.

$\boxed{\text{finite}}$ correlation functions:

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle \quad , n \in \mathbb{N}_0$$

$n=0$: $\langle 1 \rangle \overset{!}{=} 1$ normalised cor. fct.

$n=1$: $\phi(x) := \langle \varphi(x) \rangle$ mean field

$n=2$: $G(x,y) := \langle \varphi(x)\, \varphi(y) \rangle - \phi(x)\, \phi(y)$

propagator (connected 2 point fct)

$\vdots$

Generating functional: Euclidean space

finite $Z[J]$ with $\frac{1}{Z[J]}\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)} = \langle\varphi(x_1)\cdots\varphi(x_n)\rangle$
$\Uparrow$ (4.12)

$Z[J]$ is the renormalised finite generating functional of normalised Green functions (correlation fcts.) of the theory.

Reminder: classical action

$$S[\varphi] = \frac{1}{2}\int d^dx\left(\partial_\nu\varphi(x)\,\partial_\nu\varphi(x) + m^2\varphi(x)^2\right) + \frac{1}{4}\int d^dx\,\lambda\,\varphi(x)^4 \tag{4.13}$$

and

$$Z[J] = \frac{1}{N}\int[d\varphi]_{ren}\,e^{-S[\varphi]+\int d^dx\,J(x)\varphi(x)} \tag{4.14}$$

with e.g.

$$N = \int[d\varphi]_{ren}\,e^{-S[\varphi]}\quad,\ N=1$$

- In the path integral representation the task is to define $\int d\varphi \, e^{-S}$.

- $Z[J]$ generates also disconnected Green functions.

  $\Rightarrow$ Schwinger functional $W[J]$:

$$W[J] = \ln Z[J] \qquad \text{finite} \tag{4.15}$$

generates connected Green functions

proof when deriving the flows (FRG)

- $\Gamma[\phi]$ generates 1PI Green functions

$$\Gamma[\phi] = \sup_{J} \left\{ \int d^dx \, J(x)\phi(x) - W[J] \right\} \tag{4.16}$$

$$\Rightarrow \quad \phi(x) = \left.\frac{\delta W}{\delta J(x)}\right|_{J_{sup}} \qquad \text{(if differentiable)} \tag{4.17}$$

$$\frac{\delta \Gamma}{\delta \phi(x)} = \int d^dx' \, \frac{\delta J_{sup}(x')}{\delta \phi(x)} \phi(x') + J_{sup}(x) - \int d^dx' \, \frac{\delta J_{sup}(x')}{\delta \phi(x)} \cdot \left.\frac{\delta W}{\delta J}\right|_{J_{sup}}$$

$$= J_{sup}(x) \tag{4.18}$$

1PI proof with flows

$$\int d^d x' \frac{\delta^2 W[J]}{\delta J(x)\, J(x')} \quad \frac{\delta^2 \Gamma}{\delta\phi(x')\,\delta\phi(y)} \left[= \delta^{(d)}(x-y)\right]$$

$$= \int d^d x' \left(\frac{\delta}{\delta J(x)} \phi(x')\right) \frac{\delta}{\delta\phi(x')} J(y)$$

$$= \frac{\delta}{\delta J(x)} J(y) = \delta^{(d)}(x-y) \qquad (4.19)$$

or with $\Gamma^{(n)}(x_1,\dots,x_n) = \dfrac{\delta^n \Gamma}{\delta\phi(x_1)\cdots\delta\phi(x_n)}$

$$W^{(n)}(x_1,\dots,x_n) = \frac{\delta^n W}{\delta J(x_1)\cdots\delta J(x_n)}$$

$$\boxed{\int d^d x' \cdot W^{(2)}(x, x')\, \Gamma^{(2)}(x', y) = \delta^{(d)}(x-y)} \qquad (4.20)$$

and $G(x,y) = W^{(2)}(x,y) = 1/\Gamma^{(2)}(x,y)$ (4.21)

The above relations are valid in the presence of non-vanishing fields/currents, e.g.

$$\Gamma^{(2)} = \Gamma^{(2)}[\phi](x_1, x_2) \qquad (4.22)$$

- functional relations (instead of path integral)

Quantum equations of motion [Dyson-Schwinger eq. DSE]

$$\int [d\varphi]_{ren} \frac{\delta}{\delta \varphi(x)} \left\{ e^{-S[\varphi] + \int d^d x\, J(x)\, \varphi(x)} \right\} = 0$$

(4.23)

$$\Rightarrow \langle J(x) \rangle_J - \left\langle \frac{\delta S[\varphi]}{\delta \varphi(x)} \right\rangle_J = 0$$

$$\Rightarrow \boxed{J(x) = \left\langle \frac{\delta S[\varphi]}{\delta \varphi(x)} \right\rangle_J} \qquad (4.24)$$

Important relation:

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_J = \left(\frac{\delta}{\delta J(x_1)} + \phi(x_1) \right) \langle \varphi(x_2) \cdots \varphi(x_n) \rangle_J$$

(4.25)

remember: $\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_J = \frac{1}{Z[J]} \int [d\varphi]_{ren}\, \varphi(x_1) \cdots \varphi(x_n)\, e^{-S + \int J\varphi}$

$$\Rightarrow \quad \langle \varphi(x_1) \cdots \varphi(x_n) \rangle_J = \prod_{i=1}^{n} \left(\frac{\delta}{\delta J(x_i)} + \phi(x_i) \right)$$

Use
$$\frac{\delta}{\delta J(x_i)} = \int d^d x \, \frac{\delta \phi(x')}{\delta J(x_i)} \frac{\delta}{\delta \phi(x')} = \int d^d x \, G(x_i, x') \frac{\delta}{\delta \phi(x')}$$

$$= G \cdot \frac{\delta}{\delta \phi}(x_i) \qquad (4.26)$$

$$\Rightarrow \quad \boxed{\frac{\delta \Gamma}{\delta \phi(x)} = \frac{\delta S}{\delta \varphi(x)} \left[\varphi(x) = G \frac{\delta}{\delta \phi}(x) + \phi(x) \right]} \qquad (4.27)$$

S action of real scalar field:

$$\frac{\delta S}{\delta \varphi(x)} = -\partial_\mu^2 \varphi(x) + m^2 \varphi(x) + \lambda \varphi(x)^3 \qquad (4.28)$$

$$\Big|_{\varphi = G \frac{\delta}{\delta \phi} + \phi} = -\partial_\mu^2 \phi(x) + m^2 \phi(x) + \lambda \cdot \phi(x)^3$$
$$+ \lambda \left[\left(G \frac{\delta}{\delta \phi} + \phi \right)^3 - \phi^3 \right]$$

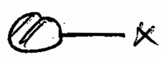
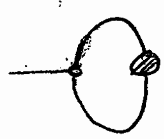
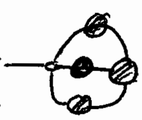
$$\Rightarrow \left.\frac{\delta S}{\delta\varphi(x)}\right|_{\varphi = G\frac{\delta}{\delta\phi}+\phi} = \frac{\delta S[\phi]}{\delta\phi(x)} + \lambda\left(G\frac{\delta}{\delta\phi}(x)\,\phi^2(x) + \phi(x)\, G\frac{\delta}{\delta\phi}(x)\,\phi(x)\right)$$

$$+ \lambda\left(G\frac{\delta}{\delta\phi}\right)^2\phi$$

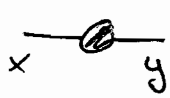
$$= \frac{\delta S[\phi]}{\delta\phi(x)} + 3\lambda\, G(x,x)\,\phi(x)$$

$$-\lambda \prod_i \int d^dx_i\, G(x,x_i)\,\Gamma^{(3)}(x_1,x_2,x_3) \tag{4.29}$$

Diagrammatically:

 $=$ $\dfrac{\delta S}{\delta\phi(x)}$ $+\dfrac{1}{2}$  $-\dfrac{1}{3!}$ 

with

 $= 1/\Gamma^{(2)}[\phi](x,y)$ (4.30)

 $= \Gamma^{(n)}[\phi](x_1,\dots,x_n)$

 $= S^{(n)}[\phi](x_1,\dots,x_n)$

General DSE (including symmetry ID's)

$$\int d\varphi \, \frac{\delta}{\delta \varphi(x)} \left\{ \Psi[\varphi] \, e^{-S[\varphi] + \int J \varphi \, d^d x} \right\} = 0$$

see 'Aspects of the FRG', chapter II

Derivation of flow equation

Heuristic idea: Kadanoff block-spinning in continuum

Define (see eq. 4.9)

$$Z_k[J] = \int [d\varphi]_{p^2 \gtrsim k^2} \, e^{-S[\varphi] + \int d^dx\, \varphi(x) J(x)} \tag{4.31}$$

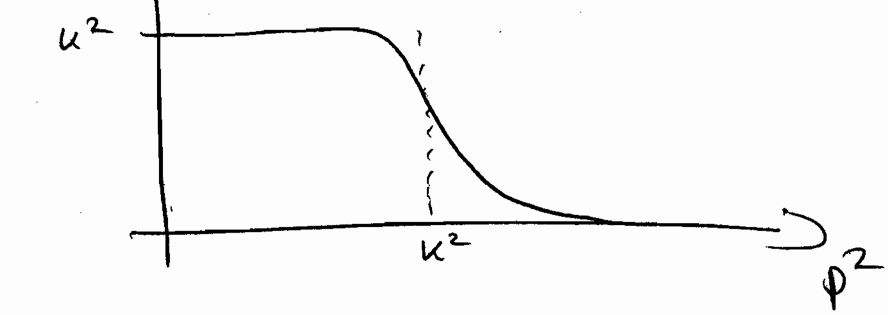
suppression of infrared (IR) modes

Practically

$$[d\varphi]_{\text{ren},\, p^2 \gtrsim k^2} = [d\varphi]_{\text{ren}} \, e^{-\Delta S_k[\varphi]} \tag{4.32}$$

with

$$\Delta S_k[\varphi] = \frac{1}{2} \int \frac{d^dp}{(2\pi)^d} \varphi(p) \, R_k(p^2) \, \varphi(-p) \tag{4.33}$$



Fourier transformation

$$\varphi(x) = \int \frac{d^d p}{(2\pi)^d} \overset{(\sim)}{\varphi}(p)\, e^{ip\cdot x} \tag{4.34}$$

$$\Rightarrow \varphi(p) = \int d^d x\, \varphi(x)\, e^{-ipx}$$

In particular:

$$\int d^d x\, d^d y\, \varphi(x)\, f(x,y)\, \varphi(y)$$

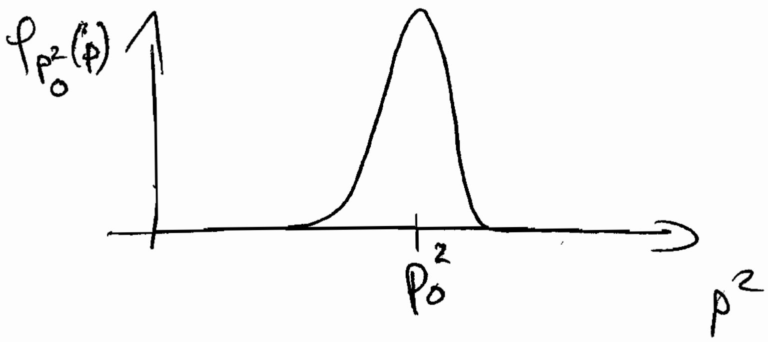
$$= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \varphi(p)\, \varphi(q) \int d^d x\, d^d y\, f(x,y)\, e^{ipx} e^{iqy}$$

$$= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \varphi(p)\, \varphi(q) \cdot f(-p,-q) \tag{4.35}$$

Regulator: $R_k(x,y) = R_k(-\partial_x^2)\, \delta^{(d)}(x-y)$

$$\Rightarrow R_k(-p,-q) = R_k(p^2)\, \delta^{(d)}(p+q)$$

$\Rightarrow$ page 116 (4.36)



$$\Delta S_k[\varphi_{p_0^2 \ll k^2}] \simeq \frac{1}{2}\left(\int d^d p \; \varphi_{p_0^2}(p)\,\varphi_{p_0^2}(-p)\right) k^2 .$$

mass-suppression

$$\Delta S_k[\varphi_{p_0^2 \gg k^2}] \simeq 0 \tag{4.37}$$

$\Rightarrow$ $\boxed{\text{IR suppressed}}$

2nd lecture

Limits:

- UV: $k \to \infty$, all momentum modes suppressed

  ΔS_k dominates S

  $\Rightarrow$ Gaussian path integral

- IR: $k \to 0$, no modes suppressed

  $\Delta S_k \to 0$

  $Z_k \to Z$

Question: $\int [d\varphi]_{ren}\, e^{-\Delta S_k[\varphi]}$

renormalised measure?

Not necessarily naively

Formally correct:

(1) $Z[J]$ finite renormalised gen. funct.

of " " Green fcts.

(2) $\frac{\delta^n Z}{\delta J^n}$ exists for all n

((1) assumes existence of theory and

(2) 'good' choice of field variable $\varphi(k)$)

(3)

$$\boxed{Z_k[J] = e^{-\Delta S_k[\frac{\delta}{\delta J}]}\, Z[J]} \tag{4.38}$$

[$\simeq e^{-\Delta S_k[\frac{\delta}{\delta J}]} \int d\varphi\, e^{-S[\varphi]+\int J\varphi}$

$= \int d\varphi\, e^{-S[\varphi]-\Delta S_k[\varphi]+\int J\varphi}$

$e^{f[\frac{\delta}{\delta J}]}\, e^{\int J\varphi} = e^{f[\varphi]}$]

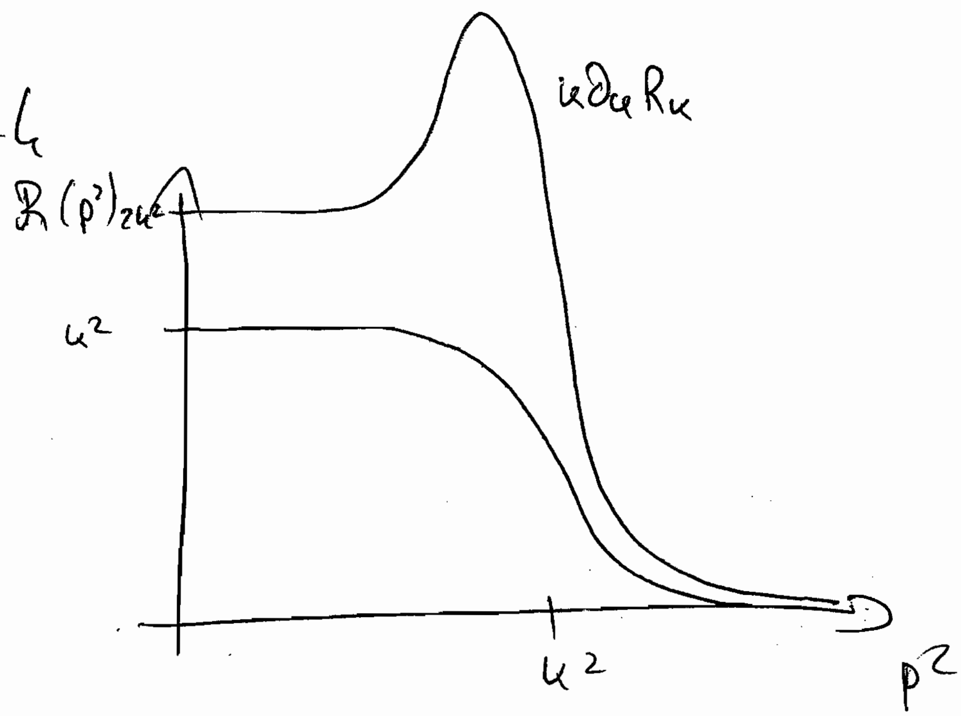
Flow equation:

$$k\,\partial_k Z_k[J] = -\left(k\partial_k \Delta S_k\left[\frac{\delta}{\delta J}\right]\right)\underbrace{e^{-\Delta S_k[\frac{\delta}{\delta J}]}Z[J]}_{Z_k[J]}$$

$$= -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\frac{\delta}{\delta J(p)}\,k\partial_k R(p^2)\,\frac{\delta}{\delta J(-p)}Z_k[J] \qquad (4.39)$$

$$\Rightarrow \boxed{k\,\partial_k Z_k[J] = -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\frac{\delta^2 Z_k[J]}{\delta J(p)\,\delta J(-p)}\,k\partial_k R_k(p^2)} \qquad (4.40)$$

with



Flow of Schwinger functional $\boxed{t = \ln k}$

- $\frac{1}{Z_k}\, k\partial_k Z_k = k\partial_k \ln Z_k = \partial_t W_k$

- $$\frac{1}{Z_k[J]} \frac{\delta^2 Z_k[J]}{\delta J(p)\,\delta J(-p)} = \frac{\delta^2 W_k}{\delta J(p)\,\delta J(-p)} + \phi(p)\,\phi(-p) \qquad (4.41)$$

with $\phi(p) = \frac{\delta W}{\delta J(p)}$

and
$$\frac{\delta^2 \ln Z_k}{\delta J(p)\,\delta J(-p)} = \frac{\delta}{\delta J(p)} \frac{1}{Z_k} \frac{\delta Z_k}{\delta J(-p)} = \frac{1}{Z} \frac{\delta^2 Z_k}{\delta J(p)\,\delta J(-p)} - \frac{1}{Z_k} \frac{\delta Z_k}{\delta J(p)} \frac{1}{Z_k} \frac{\delta Z_k}{\delta J(-p)} \qquad (4.42)$$

$\Rightarrow$
$$\boxed{\partial_t W_k[J] = -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \left[W_k^{(2)}(p,-p) + \phi(p)\phi(-p) \right] \cdot \partial_t R_k} \qquad (4.43)$$

Flow of effective action

$$\Gamma_k[\phi] = \sup_J \left\{ \int d^dx\, J(x)\phi(x) - W_k[J] \right\} - \Delta S_k[\phi] \tag{4.44}$$

Flow: $(J = J_{sup}[\phi])$

$$\partial_t\big|_\phi \Gamma_k[\phi] = \int d^dx\, \partial_t\big|_\phi J(x) \underbrace{\left[\phi(x) - \overset{\phi(x)}{\overset{\parallel}{\frac{\delta W_k[J]}{\delta J(x)}}} \right]}_{0} - \partial_t\big|_J W_k[J] - \partial_t\big|_\phi \Delta S_k[\phi]$$

$$= \frac{1}{2}\int \frac{d^dp}{(2\pi)^d} \left[W_k^{(2)}(p,-p) + \phi(p)\phi(-p) - \phi(p)\phi(-p) \right] \cdot \partial_t R_k \tag{4.45}$$

$$\Rightarrow \quad \boxed{\partial_t \Gamma_k[\phi] = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d} \left[W_k^{(2)}(p,-p) \right] \partial_t R_k(p)} \tag{4.46}$$

Relation between ϕ-der. of Γ_k

and J-der. of W:

(i) $\Gamma_k[\phi] + \Delta S_k[\phi]$ Legendre trafo of W_k

$\Rightarrow$ p. 112 – p. 113:

$$\frac{\delta(\Gamma_k + \Delta S_k)}{\delta\phi(x)} = J_{\sup_k}(x)$$

$$\frac{\delta W_k}{\delta J(x)} = \phi(x) \qquad (4.47)$$

p. 112a:

$$\int d^dx' \, \frac{\delta^2 W_k[J]}{\delta J(x)\,\delta J(x')} \, \frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\phi(x')\,\delta\phi(y)} = \delta^{(d)}(x-y) \qquad (4.48)$$

$$\boxed{\Rightarrow \int d^dx' \, W_k^{(2)}(x,x')\left(\Gamma^{(2)} + R_k\right)(x',y) = \delta^{(d)}(x-y)} \qquad (4.49)$$

with $\Delta S_k[\phi] = \frac{1}{2}\int d^dx \; \phi(x)\, R_k(x,y)\, \phi(y)$

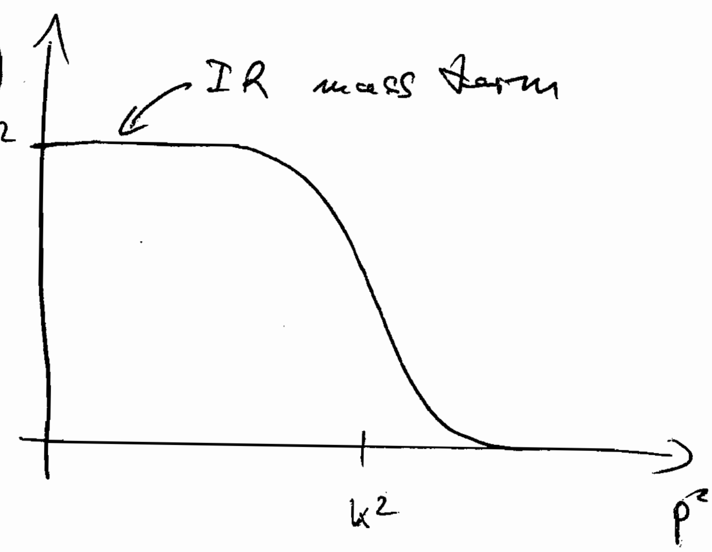
$$\Rightarrow \boxed{G_k(x,y) = W_k^{(2)}(x,y) = \frac{1}{\Gamma_k^{(2)} + R_k}(x,y)}$$

e.g. $R_k(x,y) = R_k(-\partial_x^2)\,\delta^{(d)}(x-y)$ (4.50) final flow:

$$\boxed{\partial_t \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}(p,-p)\, \partial_t R_k(p^2)} \qquad (4.51)$$

Finiteness

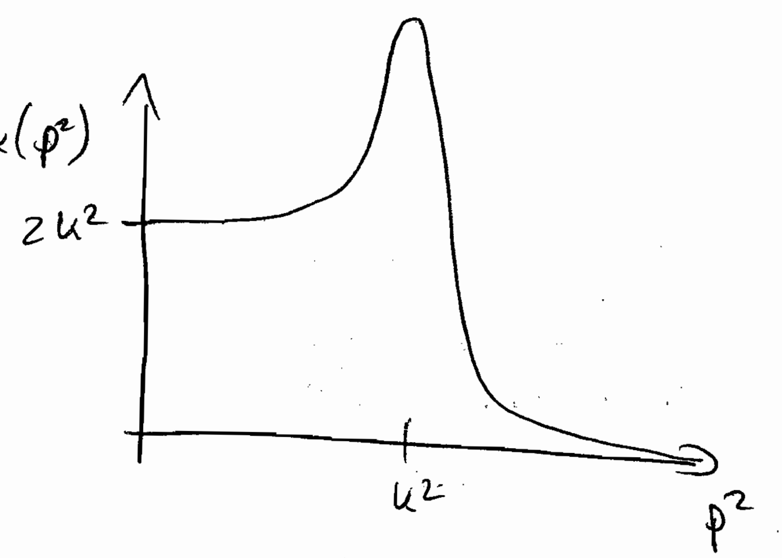
(i) IR:



$$\frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \xrightarrow{p^2/k^2 \to 0} \frac{1}{\Gamma_k^{(2)} + k^2}$$

$\Rightarrow \partial_t \Gamma_k$ IR-finite

(ii) UV:



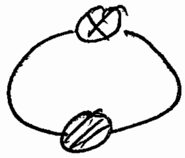
$$p^d \frac{1}{\Gamma_k^{(2)}[\phi]} R_k(p^2) \xrightarrow{p^2/k^2 \to \infty} 0$$

$\boxed{\mathrm{Tr}, \int d^d p}$

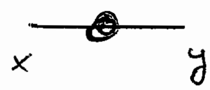
$\Rightarrow \partial_t \Gamma_k$ UV-finite

3rd lecture

Diagrammatically: (DSE, p. 115)

$\partial_t \Gamma_k = \frac{1}{2}$ 

with

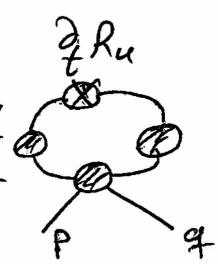


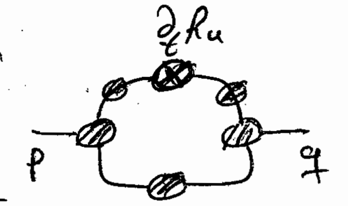
$= \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}(x,y)$

$\otimes = \partial_t R_k$

$= \Gamma^{(n)}[\phi](x_1, \dots, x_n)$

Examples:

$\frac{\delta}{\delta\phi(p)} \frac{\delta}{\delta\phi(q)} \partial_t \Gamma_k[\phi] = \partial_t \Gamma_k^{(2)}[\phi](p,q) = -\frac{1}{2}$ 

$+\frac{1}{2}\Big[$ 

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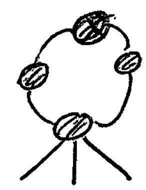
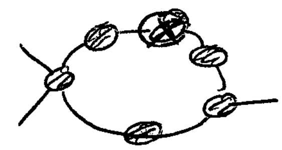
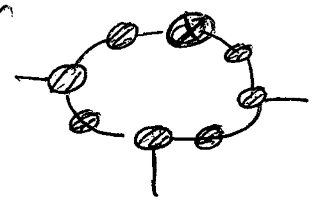
$\Big]$

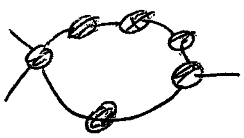
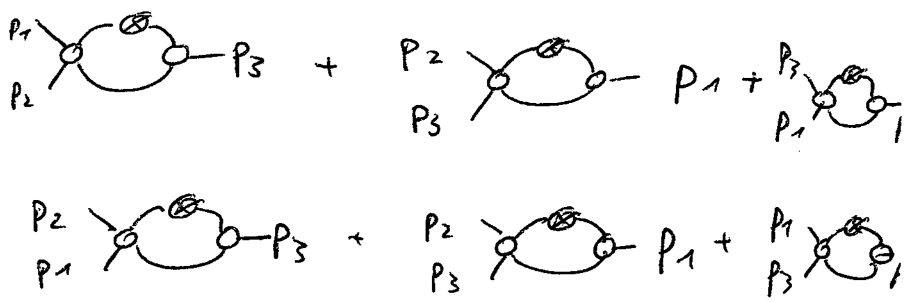
$\sim \Gamma^{(5)}$ $\qquad\qquad\qquad\qquad$ $\sim \Gamma^{(4)}$

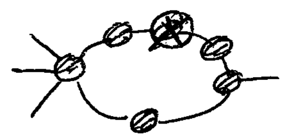
$\partial_t \Gamma^{(3)} = -\frac{1}{2}$  $+ \frac{1}{2}\Big($  $\Big)$

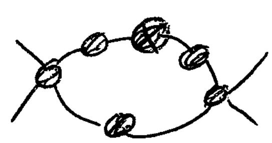
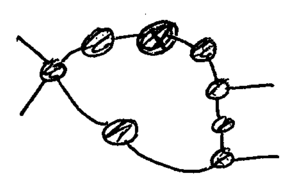
$p_1 \quad p_2 \quad p_3$

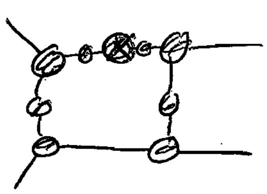
$\sim \Gamma^{(3)}$

$-\frac{1}{2}\Big($  $\Big)$ permut.

e.g.: $\Big($  $\Big) =$ 

$\partial_t \Gamma^{(4)} = -\frac{1}{2}$  $+ \frac{1}{2}\Big($  $\Big)$

$+ \frac{1}{2}\Big($  $\Big)$ $\frac{1}{2}\Big($  $\Big)$

$+ \frac{1}{2}\Big($  $\Big)$

Remarks:

(1) Finiteness, see I-14

(2) Flow equation for $\partial_t \Gamma_k[\phi]$ + initial condition $\Gamma_\Lambda[\phi]$ at some (UV/IR) scale Λ provide definition of the quantum field theory related to Γ_Λ.

(a) perturbative renormalisability

$$\lim_{\Lambda\to\infty} \Gamma_\Lambda \sim \underbrace{S_{cl}}_{\text{bare action}} \quad \text{(all perturb. relevant terms)}$$

(b) non-perturbative renormalisability

$$\lim_{\Lambda\to\infty} \Gamma_\Lambda = \Gamma_{\text{fixed-point}} \quad \text{(includes (a))}$$

(3) No restriction to momentum cut-off:

$R_k(t,t')$, $\Delta S_k \sim \int R_k(x_1,\dots,x_n)\,\phi(x_1)\dots\phi(x_n)$

time cut. Vertex/nonl. [illegible] nPI-[illegible]

'Interlude': Integro-diff. eq. for Γ_k

$$e^{-\Gamma_k[\phi]} = \int d\varphi \, e^{-(S[\varphi] + \Delta S_k[\varphi]) + \int J\cdot(\varphi-\phi)} \cdot e^{+\Delta S_k[\phi]} \tag{4.52}$$

$$\Leftrightarrow \quad e^{-\Gamma_k[\phi]} = \int d\varphi \, e^{-S[\varphi] - \Delta S_k[\varphi] + \int\left(\frac{\delta\Gamma_k}{\delta\phi} + \frac{\delta \Delta S_k}{\delta\phi}\right)(\varphi-\phi)} \cdot e^{\Delta S_k[\phi]}$$

$$= \int d\varphi \, e^{-S[\varphi] + \Delta S_k[\varphi-\phi] + \int \frac{\delta\Gamma_k}{\delta\phi}(\varphi-\phi)} \tag{4.53}$$

$$\Leftrightarrow \quad \boxed{e^{-\Gamma_k[\phi]} = \int d\varphi \, e^{-S[\varphi+\phi] + \Delta S_k[\varphi] + \int \frac{\delta\Gamma_k}{\delta\phi}\cdot\varphi}}$$

with $\boxed{\langle\varphi\rangle = 0}$ (4.54)

$k \to \infty$: $\quad \Delta S_k[\varphi] \longrightarrow \infty$: Saddle point exp. becomes exact

$$\Rightarrow \quad e^{-\Gamma_k[\phi]} \simeq e^{-S[\phi] + \text{ren.} + \mathcal{O}(1/k)} \tag{4.55}$$