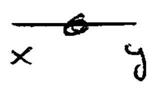
4.2 Perturbation theory

(a) 1-loop: $\partial_t \Gamma_k^{1\text{-loop}} = \frac{1}{2}$  Loop

$\Rightarrow$  $= \dfrac{1}{S^{(2)}[\phi] + R_k}(x,y)$

and hence

$$\partial_t \Gamma_k^{1\text{-loop}}[\phi] = \frac{1}{2} \operatorname{Tr} \frac{1}{S^{(2)}[\phi] + R_k} \partial_t R_k \tag{4.56}$$

k-dep

$$= \frac{1}{2} \operatorname{Tr} \partial_t \ln\left(S^{(2)}[\phi] + R_k\right)$$

($\operatorname{Tr} \partial_t \neq \partial_t \operatorname{Tr}$, strictly speaking)

Integration:

$$\Gamma_k^{1\text{-loop}}[\phi] = \Gamma_\Lambda^{1\text{-loop}}[\phi] + \int_\Lambda^k \frac{dk'}{k'} \partial_{t'} \Gamma_{k'}^{1\text{-loop}}[\phi]$$

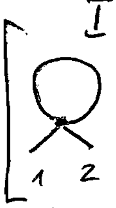
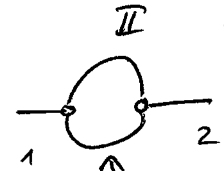
$$\Rightarrow \quad \Gamma_k^{1\text{-loop}}[\phi] = \underbrace{\Gamma_\Lambda^{1\text{-loop}}[\phi]}_{\text{renormalisation}} + \frac{1}{2}\,\mathrm{Tr}\Big[\ln\big(S^{(2)}[\phi]+R_k\big) - \ln\big(S^{(2)} + R_\Lambda\big)\Big] \qquad (4.57)$$

renormalisation

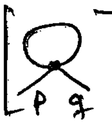
finite

e.g.:

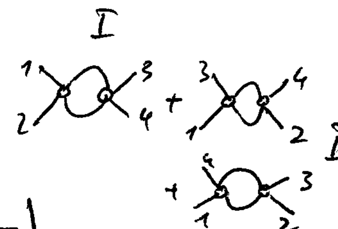
$$\Gamma_k^{1\text{-loop}\,(2)}[\phi] = \Gamma_\Lambda^{1\text{-loop}\,(2)}[\phi]$$

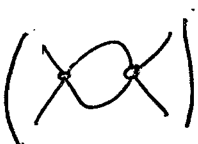
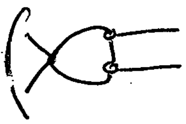
$$+ \frac{1}{2}\Big[$$  $$-$$  $$\Big]_\Lambda^k \qquad 4.58$$

$$\frac{1}{S^{(2)}[\phi]+R_k}$$

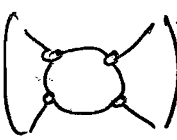
$$\Big(\Gamma_k^{1\text{-loop}}[0](p,q) = \frac{1}{2}\Big[$$  $$\Big]_\Lambda^k\Big) + \Gamma_\Lambda^{1\text{-loop}\,(2)}[0](p,q)$$

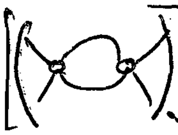
$$\Gamma_k^{1\text{-loop}\,(4)}[\phi] = \frac{\delta^2}{\delta\phi^2}\,\Gamma_k^{1\text{-loop}\,(2)}[\phi]$$



$$= \Big[-\frac{1}{2}\Big($$  $$\Big) + \frac{1}{2}\Big($$  $$\Big)$$

see pages I-74(a,b)

$$-\frac{1}{2}\Big($$  $$\Big)\Big]_\Lambda^k + \Gamma_\Lambda^{1\text{-loop}\,(4)}[\phi]$$

$$\Rightarrow \Gamma_k^{1\text{-loop}\,(4)}[0](p_1,\dots,p_4) = -\frac{1}{2}\Big[$$  $$\Big]_\Lambda^k + \Gamma_\Lambda^{1\text{-loop}\,(4)}[0](p_1,\dots,p_4)$$

Renormalisation:

(1) Γ_k indep. of Λ!

$$\Rightarrow \Lambda\partial_\Lambda \Gamma_k = 0$$

$$= \Lambda\partial_\Lambda \Gamma_\Lambda + \frac{1}{2}\mathrm{Tr}\left[\ln(\Gamma^{(2)}[\phi]+R_k)\right]_\Lambda^k \qquad (4.59)$$

(2) Λ-dep. of Γ_Λ is fixed by Flow

$\Rightarrow$ Renormalisation is

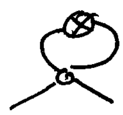
(α) adjusting Λ-indep. of Γ_k, $k\neq\Lambda$
~ regularisation

(β) fixing Λ-indep parts of Γ_Λ
~ renormalisation conditions

(3) extends trivially to full flow

Example I: m^2 in ϕ^4 - theory in 4 dim.

$$\partial_t m_k^2 = \partial^2_{\bar\phi} \; \partial_t \Gamma_k[\bar\phi \text{ const}]\Big|^{1\text{-loop}}_{\bar\phi = 0} \qquad (4.60)$$

∂_t eq. (4.58) $\rightarrow$ or p. 13a $= -\frac{1}{2}$  $+ \Big[$  $\Big]$

 $\Big|_{\phi=0} = 0 \rightarrow \; = 0$

$$\text{—} = \frac{1}{p^2 + m^2 + R_k}$$

$$\times = \lambda$$

$$\otimes = \partial_t R_k$$

$$\partial_t m_k^2 = -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2+m^2+R_k)^2}\partial_t R_k \qquad (4.61)$$

with $$S[\phi] = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, \phi(p)\,(p^2+m^2)\,\phi(-p)$$

$$+ \frac{\lambda}{4!}\int_{p_1,\dots,p_4} \phi(p_1)\circ\circ\circ\,\phi(p_4)(2\pi)^4\delta(p_1+\dots+p_4) \qquad (4.62)$$

with $$\int_p = \int \frac{d^4p}{(2\pi)^4}$$

Now we rescale all dim. full variables / parameters with $k^{-\#}$, where $\#$ is the mass dimension, i.e. g.

$$\hat{p}^2 = p^2/k^2 \quad , \quad \hat{m}^2 = m^2/k^2 \tag{4.63}$$

Hence $(\hat{k} = 1)$

$$\partial_t m_k^2 = -k^2/2 \int \frac{d^4\hat{p}}{(2\pi)^4} \frac{1}{(\hat{p}^2 + \hat{m}^2 + R_1)^2} \partial_t R_1$$

$$= -k^2/2 \int \frac{d^4\hat{p}}{(2\pi)^4} \frac{1}{(\hat{p}^2 + R_1)^2} \partial_t R_1 + \underbrace{k^2 O(\hat{m}^2)}_{O(k^0)}$$

$$= c(R_1) \cdot k^2 + O(k^0) \tag{4.64}$$

With

$$R_k = (k^2 - p^2)\,\Theta(k^2 - p^2) \tag{4.65}$$

we get

$$\partial_t m_k^2 = -k^2 \int \frac{d^4\hat{p}}{(2\pi)^4} \frac{1}{(1 + \hat{m}^2)^2} \Theta(1 - \hat{p}^2)$$

$$= \frac{k^2}{(1+\hat{m}^2)^2} \frac{\Omega_4}{(2\pi)^4} \quad , \Omega_4 = 2\pi^2 \tag{4.66}$$

See exercise

Example II: β-function in ϕ^4-theory in 4-dim

$$S[\phi] = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, \phi(p)\left(p^2+m^2\right)\phi(-p)$$

$$+\frac{\lambda}{4!}\int_{p_1,\dots,p_4} \phi(p_1)\cdots\phi(p_4)$$

$$\cdot(2\pi)^4\,\delta^{(4)}(p_1+\dots+p_4) \tag{4.67}$$

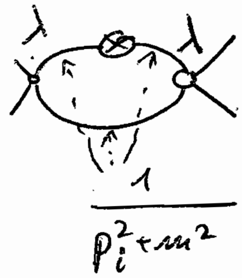
with $\int_p = \int \frac{d^dp}{(2\pi)^d}$, here $d=4$.

Inserting into $\partial_t \Gamma^{(4)}\big|_{\phi=0} = \partial_t\lambda = \dot\lambda$

1-Loop: rhs: $\Gamma^{(2)} = S^{(2)} = \left(p^2+m^2\right)\delta^{(4)}(p+q)$

$$\Gamma^{(4)} = S^{(4)} = \lambda\,(2\pi)^4\,\delta^{(4)}(p_1+\dots+p_4) \tag{4.68}$$

I-14b $\Rightarrow$ $\dot\Gamma^{(4)} = 3\,\cdot$



$$= 3\lambda^2\int\frac{d^4p}{(2\pi)^4}\left(\frac{1}{p^2+m^2+R_k}\,\dot R_k(p^2)\,\frac{1}{p^2+m^2}\right)$$

$$\cdot\frac{1}{p^2+m^2+R_k} \tag{4.69}$$

$$\overset{\circ}{\Gamma}{}^{(4)} = \overset{\circ}{\lambda} = 3\lambda^2 \cdot \int \frac{d\Omega_4}{(2\pi)^4} \cdot \frac{1}{2} \overset{\overset{dp\,p = \frac{1}{2}dp^2}{\downarrow}}{\int_0^\infty} dp^2\, p^2 \left(\frac{1}{p^2 + m^2 + R_k(p^2)}\right)^3 \overset{\circ}{R}_k(p^2)$$

(4.70)

Introduce $x = p^2/k^2$

$$R_k(p^2) = p^2\, r(x)$$

$$\Rightarrow \overset{\circ}{R}_k(p^2) = -2p^2\, x\, r'(x) = -2k^2\, x r'(x)$$

(4.71)

$$\Rightarrow \overset{\circ}{\lambda} = -\frac{2}{2}\, 3\lambda^2 \underbrace{\frac{\Omega_4}{(2\pi)^4}}_{\int d\Omega_4 = \Omega_4} \int_0^\infty dx\, x^3 \left(\frac{1}{x(1+r) + \hat{m}^2}\right)^3 r'$$

$$= -3\lambda^2 \frac{\Omega_4}{(2\pi)^4} \int_0^\infty dx \left(\frac{1}{1+r+\hat{m}^2/x}\right)^3 r'$$

$$= \frac{3}{2}\lambda^2 \frac{\Omega_4}{(2\pi)^4} \int_0^\infty dx \left[\frac{\partial}{\partial x} \frac{1}{(1+r+\hat{m}^2/x)^2} - \frac{\hat{m}^2}{x^2} \frac{1}{(1+r+\hat{m}^2/x)^3}\right]$$

(4.72)

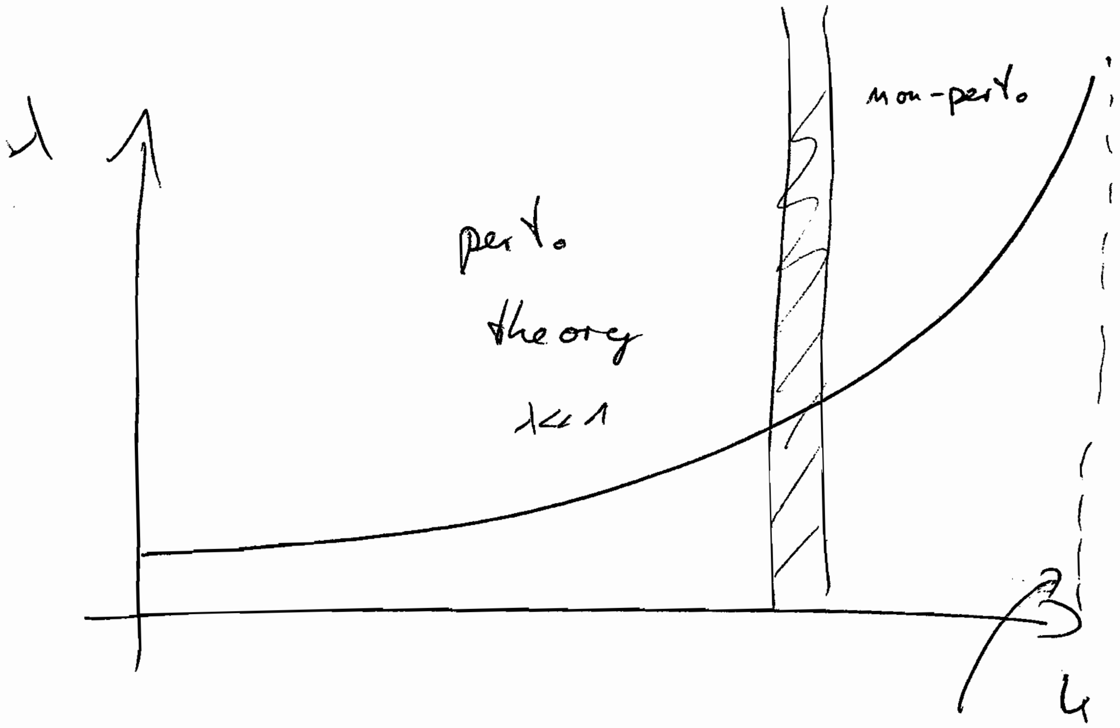
$$= \frac{3}{2}\lambda^2 \frac{\Omega_4}{(2\pi)^4} + \hat{m}^2\text{-corrections}$$

$\Omega_4 = 2\pi^2$ $\qquad\qquad \hat{m}^2 = m_0^2/k^2 + \lambda^2\text{-terms}$

(4.73)

$$\Rightarrow \boxed{\partial_t \lambda \simeq \frac{3}{16\pi^2}\lambda^2} \quad \to \lambda(t) = \frac{\lambda_0}{1 + \frac{3}{16\pi^2}(t_0 - t)\lambda_0}$$

Landau pole



Landau-pole

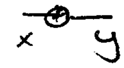
triviality: demand finite λ for $t \to \infty$

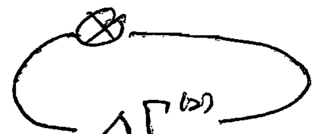
- ϕ^4-theory valid at all scales
- no Landau pole

$$\Rightarrow \boxed{\lambda_{phys} = \lambda_{k=0} \hat{=} 0} \qquad (4.74)$$

[as $t_0 \to \infty$]

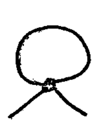
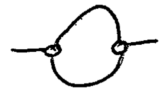
(b) 2-loop :

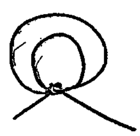
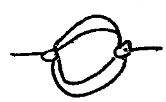
$\partial_t \Gamma_k^{2\text{-loop}} = \frac{1}{2}$  with  $= \frac{1}{\Gamma^{1\text{-loop}(2)}[\phi]+R_k}(x,$

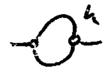
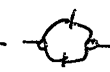
$= \frac{1}{2}$  $- \frac{1}{2}$  $\Delta\Gamma^{(2)}_{1k}$

1-loop

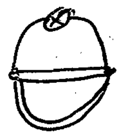
with $\Delta\Gamma^{(2)}_{1k} = \Gamma_k^{1\text{-loop}(2)} - S^{(2)}$

$= \frac{1}{2}\Big[$  $-$  $\Big]_\Lambda^k + \left(\Gamma_\Lambda^{1\text{-loop}(2)} - S^{(2)}\right)$

$= \frac{1}{2}\Big[$  $-$  $\Big]$

$= \frac{1}{2}\Big($  $-$  $\Big) - \frac{1}{2}\Big($  $-$  $\Big)$

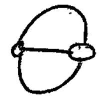
It follows

$\partial_t \Gamma_k^{2\text{-loop}} = \frac{1}{2}$  $- \frac{1}{4}$  $+ \frac{1}{4}$ 

$$\Rightarrow \partial_t \Gamma_k^{2\text{-loop}} = \partial_t(1\text{-loop}) - \frac{1}{4}\Big[$$  $$-$$  $$\Big]$$

$$+\frac{1}{4}\Big[$$  $$-$$  $$\Big]$$

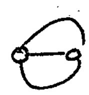
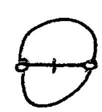
$$= \partial_t(1\text{-loop}) + \partial_t \Big\{ \frac{1}{8}$$  $$-\frac{1}{4}$$ 

$$-\frac{1}{12}$$  $$+\frac{1}{4}$$  $$\Big\}$$

$$\Rightarrow \Gamma_k^{2\text{-loop}} = \Gamma_\Lambda^{2\text{-loop}} + \int_\Lambda^k \frac{dk'}{k'}\, \partial_{t'}\Gamma_{k'}$$

$$= S_{cl} + (1\text{-loop})_{ren}$$

$$+ \frac{1}{8}$$  $$-\frac{1}{4}$$ 

$$-\frac{1}{12}$$  $$+\frac{1}{4}$$  $$+\left(\Gamma_\Lambda^{2\text{-loop}} - S - \Gamma_\Lambda^{1\text{-loop}}\right)$$

$$\Rightarrow \Delta\Gamma_k^{2\text{-loop}\,(2)} \Rightarrow 3\text{-loop}$$