4.3 Effective Potential approximation

(zeroth order derivat. expansion)

Effective Potential: Φ_c constant

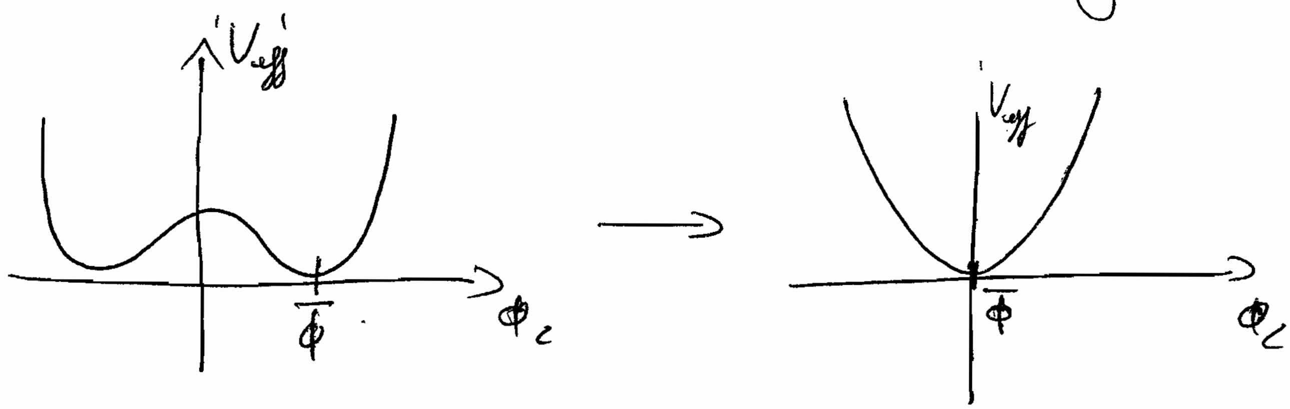
$$\mathrm{Vol}_d \cdot V_k[\Phi_c] := \Gamma_k[\Phi_c] \qquad (4.75)$$

$\uparrow \dim V_k = d$

quantum equ. of classical pot.

$$\left.\frac{\partial V_k}{\partial \Phi_c}\right|_{\bar\Phi} = 0 \quad \text{approximates ground state}$$

e.g. order parameter of symmetry breaking



broken phase

symmetric phase

$$V_{eff} = V_{k=0}$$

Examples:

(a) classical action

$$S_{cl}[\phi] = \frac{1}{2}\int d^dx\, \partial_\nu\phi\,\partial_\nu\phi + \int d^dx \left\{ \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right\} \quad (4.76)$$

$$\Rightarrow S_{cl}[\phi_c] = \left\{ \frac{m}{2}\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 \right\} \underbrace{\int d^dx}_{\mathrm{Vol}_d} \quad (4.77)$$

(b) local potential approximation (LPA)

[0th derivative expansion]

$$\Gamma_k[\phi] = \frac{1}{2}\int d^dx\, \partial_\nu\phi\,\partial_\nu\phi + \int d^dx\, V_k[\phi(x)] \quad (4.78)$$

$$\Rightarrow \boxed{\Gamma_k[\phi_c] = \mathrm{Vol}_d\, V_k[\phi_c] \quad (1.79)}$$

full flow for $V_k[\phi_c]$:

rhs requires $\Gamma_k^{(2)}[\phi_c](p,q)$:

$$\Gamma_k^{(2)}[\phi_c](p,q) = \left(Z_k(p^2,\phi_c)\,p^2 + \partial_{\phi_c}^2 V_k[\phi_c]\right)(2\pi)^d \delta^{(d)}(p+q) \qquad (4.80)$$

see p. (1.32a)

$$\Rightarrow \frac{1}{2}\int \frac{d^dp}{(2\pi)^d} \frac{1}{\Gamma_k^{(2)}[\phi_c] + R_k}(p,-p)\,\partial_t R_k(p^2)$$

see p. (1.32a)

$$= \underset{\substack{\uparrow \\ (2\pi)^d \delta^{(d)}(p+q)}}{\mathrm{Vol}_d}\, \frac{1}{2}\int \frac{d^dp}{(2\pi)^d} \frac{1}{Z_k(p^2,\phi_c)\,p^2 + \partial_{\phi_c}^2 V_k[\phi_c] + R_k(p^2)}\,\partial_t R_k(p^2) \qquad (4.81)$$

lhs: $\partial_t \Gamma_k[\phi_c] = \mathrm{Vol}_d \cdot \partial_t V_k[\phi_c]$

$$\Rightarrow \boxed{\partial_t V_k[\phi_c] = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d} \frac{1}{Z_k(p^2,\phi_c)\,p^2 + \partial_{\phi_c}^2 V_k[\phi_c] + R_k(p^2)}\,\partial_t R_k} \qquad (4.82)$$

full flow, not closed

because of Z_k

$$\Gamma^{(2)}_k[\phi_c](p,q) = \left(Z_k(p^2,\phi_c)\,p^2 + \partial^2_{\phi_c} V_k[\phi_c]\right)(2\pi)^d\,\delta^{(d)}(x-y) \tag{4.83}$$

e.g. from $\Gamma_k[\phi] = \frac{1}{2}\int d^dx\ Z_k(-\partial^2,\phi(x))\partial_\mu\phi(x)\,\partial_\mu\phi(x)$

$$+ \int d^dx\ V_k[\phi(x)]$$

with

$$\left.\frac{\delta^2\Gamma_k}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=\phi_c} = -Z_k(-\partial^2_x,\phi_c)\ \partial^2_x\ \delta(x-y)$$

$$+ \partial^2_{\phi_c} V_k[\phi_c]\ \delta^{(d)}(x-y) \tag{4.84}$$

$$\frac{1}{\Gamma^{(2)}_k[\phi_c]+R_k}(p,q) = \frac{1}{p^2 Z_k(p^2,\phi_c) + V_k[\phi_c] + R_k(p^2)}\,\delta^{(d)}(p+q) \tag{4.85}$$

$$R_k(p,q) = R_k(p^2)\,\delta^{(d)}(p+q)$$

$$\mathring{R}_k(p,q) = \mathring{R}_k(p^2)\,\delta^{(d)}(p+q)$$

$$(2\pi)^d\,\delta^{(d)}(p=0) = \int d^dx\ e^{ipx}\Big|_{p=0} = Vol_d$$

0th order deriv. expansion:

$$Z_k(p^2, \phi_c) = 1 \quad \leftarrow \text{flow closed}$$

(4.86)

- good low energy (momentum) approximation

$\Leftrightarrow$ requires mass-scales!

regulator choice: (see p. 127b)

$$R_k = (k^2 - p^2)\,\Theta(k^2 - p^2)$$

$$\dot{R}_k = 2k^2\,\Theta(k^2 - p^2)$$

optimised cut-off

! for 0th order der. expans.!

(4.87)

$$\Rightarrow \boxed{\partial_t V_k[\phi_c] = \underbrace{\frac{\Omega_d}{(2\pi)^d}}_{\int d\Omega_d} \cdot \int_0^k dp\, p^{d-1} \frac{k^2}{k^2 + \partial_{\phi_c}^2 V_k} = \frac{1}{d} \cdot \frac{\Omega_d}{(2\pi)^d} \frac{k^{d+2}}{k^2 + \partial_{\phi_c}^2 V_k[\phi_c]}}$$

(4.88)

with

$$\Omega_d = 2\pi^{d/2} / \Gamma[d/2] \qquad (4.89)$$

Example: flow of λ in $d=4$:

$$V_k = \frac{1}{2} m_k^2 \phi_c^2 + \frac{\lambda_k}{4!} \phi_c^4 + \frac{\lambda_{6k}}{6!} \phi_c^6 + \dots$$

$$\Rightarrow \partial_{\phi_c}^2 V_k = m_k^2 + \frac{\lambda_k}{2} \phi_c^2 + \frac{\lambda_{6k}}{4!} \phi_c^4 + \dots$$

$$\partial_{\phi_c}^4 \dot{V}_k \Big|_{\phi_c=0} = \dot{\lambda}_k = \partial_{\phi_c}^4 \Big|_{\phi_c=0} \frac{1}{2} \frac{1}{16\pi^2} \frac{k^6}{\left(k^2 + m_k^2 + \frac{\lambda_k}{2}\phi_c^2 + \frac{\lambda_{6k}}{4!}\phi_c^4 + \dots\right)}$$

$$= \frac{6}{2} \frac{1}{16\pi^2} \lambda_k^2 \frac{1}{(1+\underbrace{\hat{m}_k^2}_{= m_k^2/k^2})^3}$$

$$- \frac{1}{2} \frac{1}{16\pi^2} \lambda_{6k} k^2 \frac{1}{(1+\hat{m}_k^2)^2} \qquad (4.90)$$

$\hat{m}_k^2 \sim 0$, $k^2 \lambda_{6k} \sim 0$:

$$\boxed{\dot{\lambda}_k = 3 \frac{1}{16\pi^2} \lambda_k^2} \qquad (4.91)$$

see page 127d, eq. (4.73) pert. theory

Again we resort to dim. less variables.

We rewrite eq. (4.88), p. 133 as

$$\frac{1}{k^d}\,\partial_t V_k = \frac{1}{d}\,\frac{\Omega_d}{(2\pi)^d}\,\frac{1}{1+\partial_\phi^2 V/k^2} \tag{4.92}$$

and introduce dim. less fields $\hat{\phi}$ & couplings:

$$\hat{V}(\hat{\rho}) = \frac{1}{k^d}\,V(\rho) \quad \text{with } \rho = \phi^2/2$$

$$\hat{\rho} = \rho\cdot k^{2-d} \quad \text{with } \hat{\phi} = \phi\cdot k^{(2-d)/2} \tag{4.93}$$

and hence $\left(\partial_{\hat\phi}^2 = \partial_{\hat\rho} + 2\hat\rho\,\partial_{\hat\rho}^2,\ \hat{V}' = \partial_{\hat\rho}\hat{V}\right)$

$$\frac{1}{k^d}\,\partial_t|_\rho\,(\hat{V}\,k^d) = \frac{1}{d}\,\frac{\Omega_d}{(2\pi)^d}\,\frac{1}{1+\hat{V}'+2\hat{\rho}\hat{V}''}$$

$$\Rightarrow \boxed{\left(\partial_t|_{\hat\rho} + d + (2-d)\,\hat{\rho}\,\partial_{\hat\rho}\right)\hat{V} = \frac{1}{d}\,\frac{\Omega_d}{(2\pi)^d}\,\frac{1}{1+\hat{V}'+2\hat{\rho}\hat{V}''}} \tag{4.94}$$

Exercice: How does eq. (4.94) generalise to the $O(N)$ case: ϕ^a, $a=1,\dots,N$; $V = V[\phi^a\phi^a/2]$

The fixed point equation follows from eq. (4.94) as

$$\boxed{\left(d + (2-d)\,\hat{\rho}\,\partial_{\hat{\rho}}\right)\hat{V} = \frac{1}{d}\frac{\Omega_d}{(2\pi)^d}\frac{1}{1+\hat{V}'+2\hat{\rho}\hat{V}''}} \qquad (4.95)$$

Eq. (4.95) can only be solved numerically, for a program (Mathematica) see the webpage of the lecture: (http://www.thphys.uni-heidelberg.de/critical/critical-uebungen.php).

Here we resort to the simplest approximation

$$V = \frac{1}{2}\lambda(\rho - \kappa)^2 \qquad (4.96a)$$

with

$$\partial_t V(\rho) = \frac{1}{2}\dot{\lambda}(\rho-\kappa)^2 - \lambda\dot{\kappa}(\rho-\kappa)$$

and $\qquad (4.96b)$

$$V' + 2\rho V'' = \lambda(\rho-\kappa) + 2\lambda\rho$$

In eq. (4.96) we have <u>re-named</u> the dim. less quantities with unhatted variables.

The flows for λ, κ are derived from the Taylor expansion of the flow eq. at $V'(\rho_0)=$ Note that the projection onto the flows of the couplings is <u>not</u> unique, e.g. we could also evaluate the flow at other expansion points with or within a Taylor expansion.

On the lhs this leads to ($\kappa>0$):

$$\partial_\rho^2\Big|_{\rho=\kappa}\left(\partial_t+d+(2-d)\rho\,\partial_\rho\right)V\Big|_{\rho=\kappa}=\underbrace{\beta_\lambda}_{\dot\lambda}-(4-d)\lambda$$

$$\partial_\rho\Big|_{\rho=\kappa}\left(\partial_t+d+(2-d)\rho\,\partial_\rho\right)V\Big|_{\rho=\kappa}=-\lambda\underbrace{\beta_\kappa}_{\dot\kappa}-(2-d)\kappa\lambda \qquad (4.97)$$

rhs:

$$\partial_\rho^2\Big|_{\rho=\kappa}\frac{1}{d}\frac{\Omega_d}{(2\pi)^d}\frac{1}{1+V'+2\rho V''}\Big|_{\rho=\kappa}=\frac{18}{d}\frac{\Omega_d}{(2\pi)^d}\lambda^2\frac{1}{(1+2\lambda\kappa)^2}$$

(4.98)

$$\partial_\rho\Big|_{\rho=\kappa}\frac{1}{d}\frac{\Omega_d}{(2\pi)^d}\frac{1}{1+V'+2\rho V''}\Big|_{\rho=\kappa}=-\frac{3}{d}\frac{\Omega_d}{(2\pi)^d}\lambda\frac{1}{(1+2\lambda\kappa)^2}$$

Finally

$$\boxed{\begin{aligned} \beta_\lambda &= (d-4)\lambda + \frac{18}{d}\frac{\Omega_d}{(2\pi)^d}\lambda^2 \frac{1}{(1+2\lambda k)^3} \\ \beta_k &= (2-d)\,k + \frac{3}{d}\frac{\Omega_d}{(2\pi)^d}\frac{1}{(1+2\lambda k)^2} \end{aligned}} \qquad (4.99)$$

Fixed points: $\boxed{\vec{\beta}_* = \begin{pmatrix}\beta_k \\ \beta_\lambda\end{pmatrix} = 0}$

(i) Gaußian FP:

$$(k_*, \lambda_*) = (0,0) \qquad (4.100)$$

Note that $\beta_{k_*} \neq 0$ but the flow for k was $\lambda \beta_k = \lambda(\dots)$, see eq. (4.97)/(4.98)

(ii) Wilson-Fisher FP in 3d:

$$\Omega_d = \frac{2\pi^{3/2}}{\underbrace{\Gamma(3/2)}_{=\sqrt{\pi}/2}} = 4\pi$$

$$\Omega_d/(2\pi)^d = \frac{1}{2\pi^2} \qquad (4.101)$$

The β-fcts. are given by

$$\boxed{\begin{aligned} \beta_\lambda &= -\lambda + \frac{3}{\alpha^2}\,\lambda^2\,\frac{1}{(1+2\lambda k)^3} \\ \beta_k &= -k + \frac{1}{2\alpha^2}\,\frac{1}{(1+2\lambda k)^2} \end{aligned}} \qquad (4.102)$$

The stability matrix reads $(\vec{\lambda} = (k, \lambda))$

$$B_{ij} = \frac{\partial \beta_i}{\partial \lambda_j} = \begin{pmatrix} -1 - \frac{2}{\alpha^2}\frac{\lambda}{(1+2k\lambda)^2} & -\frac{18}{\alpha^2}\frac{\lambda^3}{(1+2k\lambda)^4} \\ -\frac{2}{\alpha^2}\frac{k}{(1+2k\lambda)^3} & -1 + \frac{6}{\alpha^2}\frac{\lambda(1-k\lambda)}{(1+2k\lambda)^4} \end{pmatrix} \qquad (4.103)$$

The eigenvalues ω_i $B \cdot \delta\lambda_i = \omega_i \delta\lambda_i$, $i = 0, \ldots$, follow

$$\omega_0 = -2, \qquad \omega_1 = 1/3 \qquad (4.104)$$

and hence we have

$$\boxed{\nu = 1/2, \quad \omega_1 = -b_2 = 1/3} \qquad (4.105)$$

$\nu = -1/\omega_0$ ← see p. 89, eq. (3.64b)

(note the opposite scaling $\lambda \to 1/\lambda$)

Now we increase the polynomial truncation,

and with

$$V_k[\rho] = \sum_{n=1}^{N_{max}=6} \frac{\lambda_n}{n!}(\rho - \kappa)^n \tag{4.106}$$

we get

$$\nu = 0.647 \quad , \quad \omega = 0.672 \tag{4.107}$$

as compared to the result for $N_{max} \to \infty$.

$$\boxed{\nu = 0.65 \quad , \quad \omega = 0.656} \tag{4.108}$$

The theoretical value for ν is

$$\nu = 0.63\ldots \tag{4.109}$$

Remarks:

(i) The value for ν, eq. (4.108) depends on the regulator. The regulator used, eq. (4.87), p. 133, is optimised and provides the value for ν closest to the exact one.

The range of ν's is given by

$$\boxed{0.65 \leq \nu \leq 0.69} \qquad (4.110)$$

$$\begin{array}{cc} R_{opt} & R_{sharp} \\ \| & \| \\ (k^2-p^2)\,\Theta(k^2-p^2) & p^2\left(\frac{1}{\Theta(p^2-k^2)} - 1\right) \end{array}$$

see also script of 'Non-pert. methods in gauge theories' SS 08.

(ii) The final step towards the exact result is the inclusion of the <u>wave function renormalisation</u> $Z_k(p^2, \phi_c)$, eq. (4.80), p. 132. If the full Z_k could be computed, the flow for V_k would be exact. However, the flow of Z_k requires more than the knowledge of Z_k & V_k.

(iii) The next step leading to a closed set of equation is the approximation to $\Gamma_k^{(2)}$,

$$\boxed{\Gamma_k^{(2)}[\phi_c](p,q) = \left(Z_k\, p^2 + \partial_{\phi_c}^2 V_k(\phi_c)\right) \cdot (2\pi)^d\, \delta(p+q)} \tag{4.111}$$

The flow of Z_k is extracted from

$$\partial_{p^2}\Big|_{p=0} \dot{\Gamma}_k^{(2)}[\phi_c](p,q) \tag{4.112}$$

and leads to

$$\begin{aligned} \nu &= 0.64\ldots \\ \eta &= 0.044\ldots \end{aligned} \tag{4.113}$$

in comparison to $\nu = 0.63\ldots$, $\eta = 0.034\ldots$