5 O(N) - theories & critical phenomena

In this chapter we put to work the concepts & tools we have collected so far. We shall utilise that at a 2^{nd} order phase transition physics is governed by the universal aspects of the theory; its symmetries. The O(N)-symmetric scalar theories hence give us access to a wide range of critical phenomena having an O(N)-symmetry.

Examples:

3d: $N = 0$: Statistical prop. of polymer chains (self-avoiding walks) $[N \to 0]$

Ising $N = 1$ $\mathbb{Z}_2$: • liquid-vapor transition, uniaxial

discrete • antiferromagn. materials,

• Conf-deconf in SU(2) (SU(3): $\mathbb{Z}_3$)

⋮

XY

$N=2$: • Helium (^4He) superfluid transition

U(1)

• (anti) ferromagnets with easy-plane anisotropy (XY)

⋮

Heisenberg

$N=3$: • Curie transition in isotropic ferromagnets

O(3)

• antiferromagnets at the Néel transition point

⋮

$N=4$: • chiral phase transit. with 2 light quarks in QCD

• electroweak theory with vanish. gauge and Yukawa couplings

⋮

$N\to\infty$: $\frac{1}{N}$ expansion for comp. at finite N

2d: $N=2$: Kosterlitz-Thouless

5.1 O(N)-symmetric scalar theories

We extend the Ising-type scalar model with one real scalar field ϕ to a scalar theory with N real scalar fields,

$$\phi = \begin{pmatrix} \phi^1 \\ \vdots \\ \phi^N \end{pmatrix}, \qquad (5.1)$$

and an action $S[\phi]$ which is symmetric under ~~global~~ O(N)-transfo: $\mathcal{O} \in O(N)$

$$\phi \to \phi^{\mathcal{O}} = \mathcal{O}\,\phi\,, \qquad (5.2)$$

$$S[\phi] \to S[\phi^{\mathcal{O}}] \overset{!}{=} S[\phi]\,,$$

The classical action has the general form

$$S[\phi] = \frac{1}{2}\int d^dx\, \partial_\mu \phi^a \partial_\mu \phi^a + \int d^dx\, V(\rho)$$

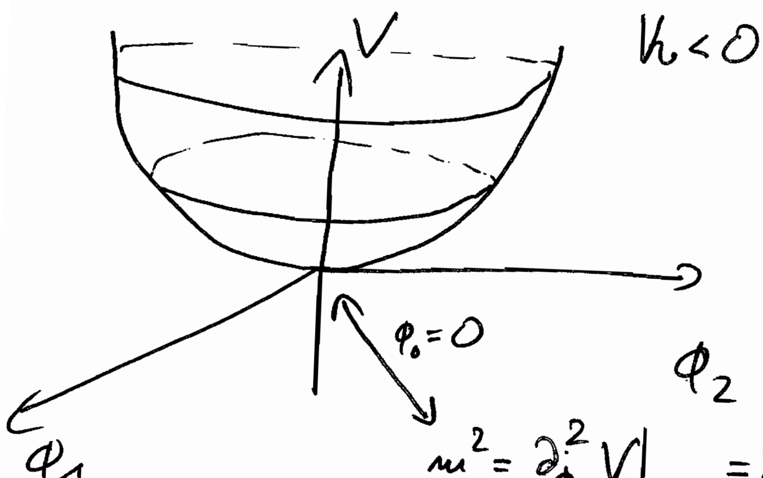
with $\qquad (5.3)$

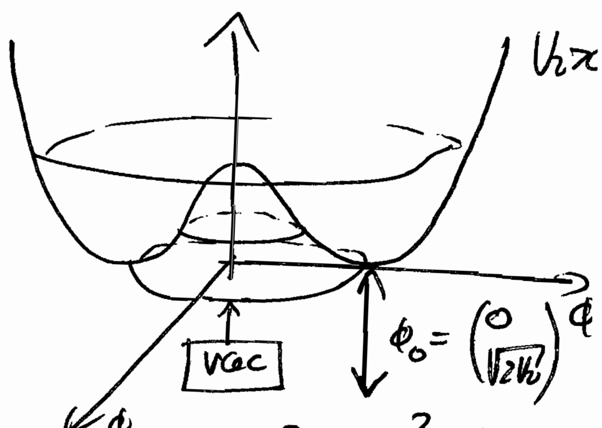
$$\rho = \frac{1}{2}\phi^a\phi^a$$

Renormalisability in $d \leq 4$ enforces

$$V(\rho) = \frac{1}{2}\lambda(\rho - \kappa)^2 \qquad (5.4)$$

and no higher term $\rho^n, n>2$. The action eq. (5.3) very much resembles the Ising model action in its form used within the flow eq., see e.g. p. 135, eq. (4.93). Here, however, we have N propagating dof's and for $N \geq 2$ we have a continuous symmetry. For example, for $N=2$ we have





$$m^2 = \partial^2_{\phi_1} V\big|_{\phi=0} = \partial^2_{\phi_2} V\big|_{\phi=0} = \partial_\rho V + 2\rho\,\partial_\rho^2 V = -\lambda k$$

$$m_2^2 = \partial^2_{\Phi_2} V\big|_{\Phi_2^2/2 = k} = 2\lambda k$$

$$m_1^2 = \partial^2_{\Phi_1} V\big|_{\Phi_2^2/2 = k} = 0$$

massless mode $\longrightarrow$ $m_1^2 = 0$

$$\boxed{\text{vac} \simeq O(2)} \to \text{broken to } O(1) \qquad (5.5)$$

For general $O(N)$-symmetry we have $(N-1)$ massless modes with an $O(N-1)$ symmetry. To see this we parametrise the scalar field as

$$\phi = \mathcal{O} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{2\rho} \end{pmatrix} \qquad (5.6)$$

with $\sqrt{2\rho} = \nu_0 + \sigma$. Evidently, $\mathcal{O} \in O(N-1) \subseteq O(N)$ with

$$\mathcal{O} = \begin{pmatrix} \mathcal{R} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{R} \in O(N-1) \qquad (5.7)$$

leaves invariant the vacuum vector $(0, \dots, 0, \nu_0)$. This also entails that, $\phi_0 = (0, \dots, 0, \sqrt{2\nu_0})$,

$$\partial_{\phi^a} \partial_{\phi^b} V(\rho)\Big|_{\phi_0} \quad 2\lambda\nu_0\, \delta^{aN} \delta^{bN} \qquad (5.8)$$

We choose the parameterisation

$$\phi = \begin{pmatrix} \vec{\pi} \\ \sqrt{2\nu_0} + \sigma \end{pmatrix} \qquad (5.9)$$

where σ is the massive radial mode and π_i, $i = 1, \dots, N-1$ are the massless Goldstone modes.

For $N=4$, related to QCD with 2 (light) flavours σ relates to the σ-meson, whereas $\vec{\pi}$ relates to the pions, the (pseudo) Goldstone bosons of strong chiral sym. breaking. With eq. (5.9), the action eq. (5.3) takes the form, $\rho-h=\sigma^2+\sqrt{2h}\sigma+\vec{\pi}^2$;

$$\boxed{\begin{aligned} S[\sigma,\vec{\pi}] = \int d^dx \Big\{ & \tfrac{1}{2}\,\sigma\left(-\partial_\nu^2+2h\lambda\right)\sigma + \tfrac{1}{2}\,\vec{\pi}\left(-\partial_\nu^2\right)\vec{\pi} \\ & + \tfrac{1}{8}\left(\sigma^2+\vec{\pi}^2\right)^2 + \frac{\lambda\sqrt{2h}}{2}\,\sigma\left(\sigma^2+\vec{\pi}^2\right) \end{aligned}} \quad (5.10)$$

Note that eq. (5.10) still is $O(N)$-symmetric! This symmetry is hidden, but the $O(N-1)$ symmetry

$$\vec{\pi} \to \vec{\pi}^R = R\,\vec{\pi}\,, \quad R \in O(N-1), \quad (5.11)$$

for the <u>massless</u> π_i-fields, $i=1,\dots,N-1$, is apparent: $\vec{\pi}^2 \to \vec{\pi}^{R\,2} = \vec{\pi}^2$.

'For every spontaneously broken continuous symmetry the theory must contain a massless particle, the Goldstone boson'

Given an action

$$S[\phi] = S_0[\partial\phi] + \underbrace{\mathrm{Vol}_d}_{\int d^dx} V(\phi), \tag{5.12}$$

with continuous symmetry (infinitesimal)

$$\phi \to \phi^{\Omega} = \phi + \varepsilon\,\Omega(\phi) \; : \; \boxed{S[\phi^{\Omega}] = S[\phi]} \tag{5.13}$$

and minimum ϕ_0 with (ϕ_0 is constant)

$$\left.\frac{\partial V}{\partial \phi}\right|_{\phi_0} = 0 . \tag{5.14}$$

In our $O(N)$-case we have $\phi^a \to \phi^a + \varepsilon\,\Omega^a(\phi)$ with

$$\Omega^a(\phi) = \omega^{ab}\phi^b , \quad \omega \in o(N) \tag{5.15}$$

where $o(N)$ is the Lie algebra of $O(N) \ni \mathcal{O} = e^{\varepsilon\omega}$, with anti-symmetric $\omega^{ab} = -\omega^{ba}$.

In order ε the symmetry eq. (5.13) entails that

$$\boxed{\Omega^a(\phi)\frac{\partial V}{\partial \phi^a} = 0.} \tag{5.16}$$

Eq. (5.16) also leads to $\frac{\partial}{\partial \phi^b}\left(\Omega^a(\phi)\frac{\partial V}{\partial \phi^a}\right)\Big|_{\phi=\phi_0} = 0$, or

$$\frac{\partial \Omega^a}{\partial \phi^b}\Big|_{\phi_0} \cdot \underbrace{\frac{\partial V}{\partial \phi^a}\Big|_{\phi_0}}_{0 \leftarrow \text{EoM } (5.14)} + \Omega^a\Big|_{\phi_0}\frac{\partial^2 V}{\partial \phi^a \partial \phi^b}\Big|_{\phi_0} = 0. \tag{5.17}$$

This leads to

$$\boxed{\Omega^a(\phi_0)\cdot\frac{\partial^2 V}{\partial \phi^a \partial \phi^b}(\phi_0) = 0} \tag{5.18}$$

(1) Eq. (5.18) is satisfied for all possible choices of $\Omega^a(\phi_0)$. We conclude that the Hessian $\frac{\partial^2 V}{\partial \phi^a \partial \phi^b}$ has as many <u>zero eigenvalue</u> as the dimension of span $\Omega(\phi_0)$.

(2) Those Ω-choices with $\Omega(\phi_0) = 0$ are symmetry transformations that lead the vacuum intact.

We conclude that

$$\boxed{\#\text{ of Goldstones} = \#\text{ of broken symmetries}} \tag{5.19}$$

This is the Goldstone theorem on p. 170.

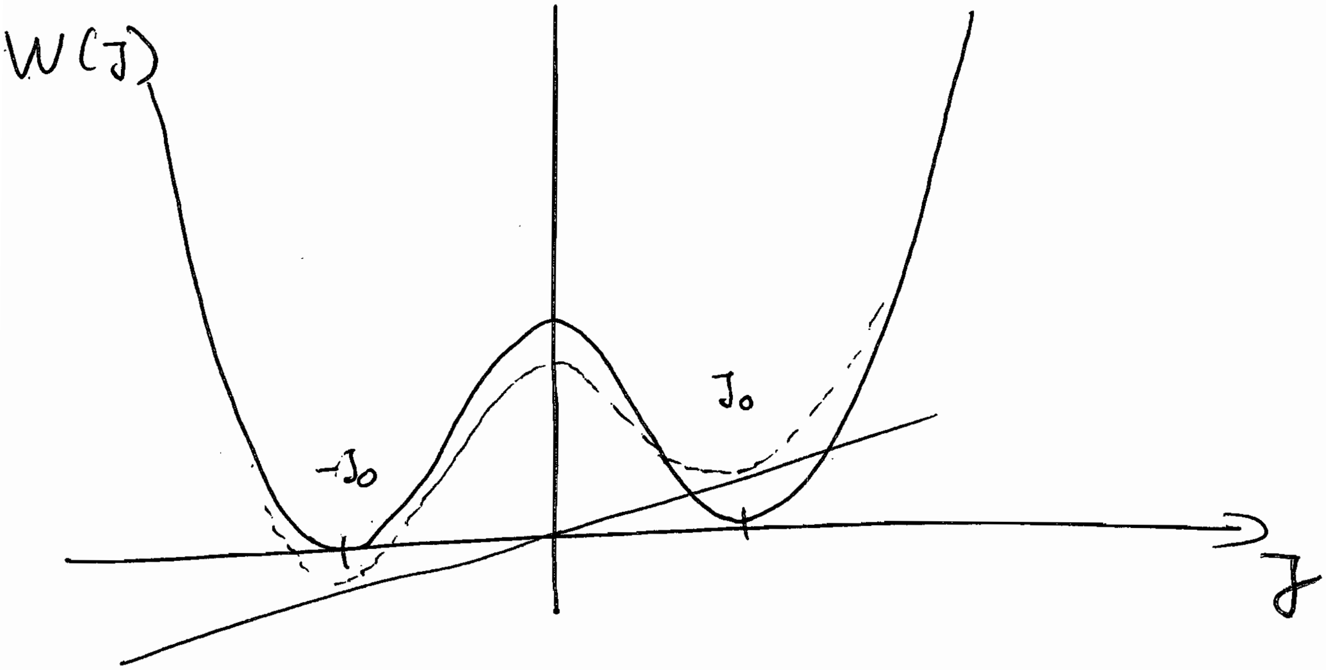
On the quantum level we simply start with the full effective action,

$$\Gamma[\phi] = \Gamma_0[\partial\phi, \phi] + \mathrm{Vol}_d\, V_{eff}[\phi] \tag{5.20}$$

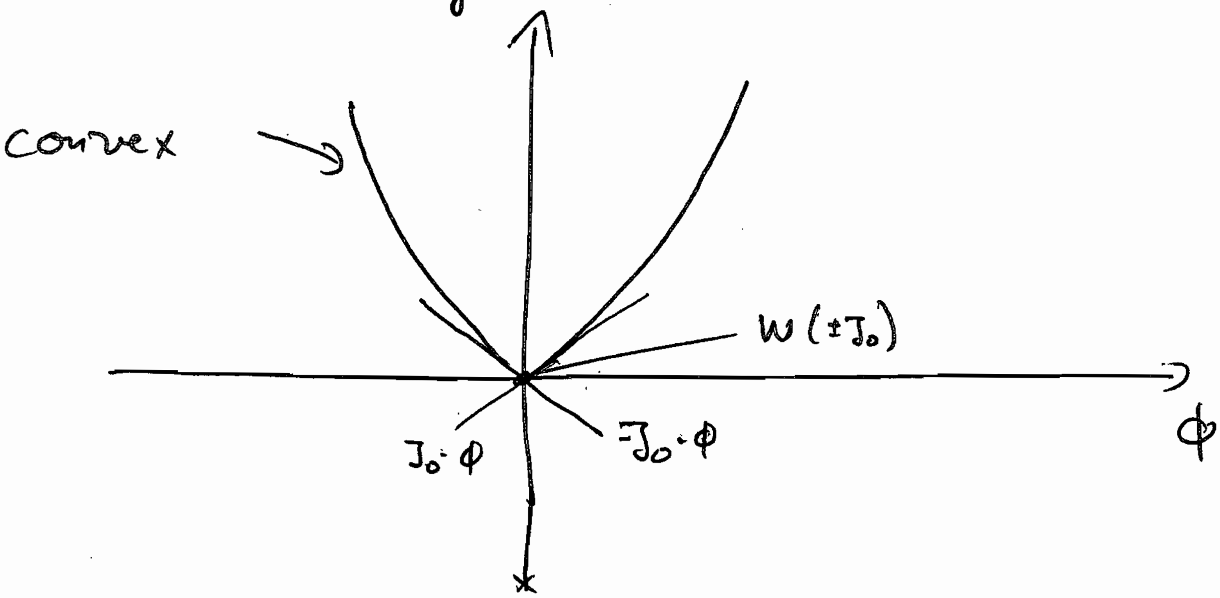
with $\Gamma_0[0,\phi] = 0$. The EoM for constant fields reads

$$\left.\frac{\partial V_{eff}}{\partial \phi}\right|_{\phi_0} = 0 \tag{5.21}$$

analagously to eq. (5.14), the starting point of classical considerations. Hence we are led to the same conclusions eq. (5.18).



$$V(\phi) = \sup_{J} \left(\phi \cdot J - W(J) \right)$$



$$\tilde{W}(J) = \sup_{\phi} \left(J \cdot \phi - V(\phi) \right)$$