5.2 Functional RG for the O(N)-model

For a first computation of the critical exponents of the O(N)-model we extend the FRG-formulation for the Ising-model (O(1), chapter 4.3) to the general case.

Again we first resort to the Effective pot. approximation, to wit,

$$\partial_t V_k[\rho] = \frac{1}{2} \int \frac{d^dp}{(2\pi)^d} \left[\frac{1}{p^2 \mathbb{1} + \frac{\partial^2 V}{\partial \phi^2} + R_k \mathbb{1}} \right]^{aa} \dot{R}_k \tag{5.22}$$

where we have used $Z(p^2, \phi) = 1$, and a diagonal regulator,

$$\Delta S_k[\phi] = \frac{1}{2} \int \frac{d^dp}{(2\pi)^d} \phi^a(p) R_k(p^2) \phi^a(-p) \tag{5.23}$$

Eq. (5.22) is a flow equation for $V(\rho)$ and hence we put $\vec{\pi} = 0$ on the rhs.

For $\vec{\pi}$ the Hessian $\frac{\partial^2 V}{\partial \phi^2}$ reads

$$\boxed{\left(\frac{\partial^2 V}{\partial \phi^2}\right)^{ab} = \partial_\rho V\, \delta^{ab} + 2\delta^{aN}\delta^{bN}\rho\, \partial_\rho^2 V. \quad (5.24)}$$

where we have used that

$$\partial_{\phi^a}\partial_{\phi^b} V(\rho) = (\partial_{\phi^a}\rho)(\partial_{\phi^b}\rho)\,\partial_\rho^2 V + (\partial_{\phi^a}\partial_{\phi^b}\rho)\,\partial_\rho V$$
$$= \phi^a\phi^b\,\partial_\rho^2 V + \delta^{ab}\partial_\rho V. \qquad (5.25)$$

At $\phi = \begin{pmatrix} 0 \\ \sqrt{2\rho} \end{pmatrix}$, eq. (5.25) reduces to the rhs of eq. (5.24)

Eq. (5.24) is easily inverted, as is $\left(p^2 + R_k(p^2)\right)\delta^{ab} + \frac{\partial^2 V}{\partial\phi^a\partial\phi^b}$

This leads to

$$\left[\frac{1}{p^2 \mathbb{1} + \frac{\partial^2 V}{\partial \phi^2} + R_k \mathbb{1}}\right]^{ab}_{\vec{\pi}=0} = \frac{1}{p^2 + R_k + \partial_\rho V}\left(\delta^{ab} - \delta^{aN}\delta^{bN}\right)$$
$$+ \frac{1}{p^2 + R_k + \partial_\rho V + 2\rho\,\partial_\rho^2 V}\,\delta^{aN}\delta^{bN} \qquad (5.26)$$

Goldstones

radial mode

This facilitates the comp. of the flow eq. (5.22), and leads to counting factors in front of the

Goldstone terms, $(N-1)$, and that of the radical mode, 1.

$$\boxed{\partial_t V_k(\rho) = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\, \dot{R}_k(p^2) \left\{ \overbrace{\frac{N-1}{p^2+R_k(p^2)+V'(\rho)}}^{\text{Goldstone modes}} + \overbrace{\frac{1}{p^2+R_k(p^2)+V'+2\rho V''}}^{\text{Radial mode}} \right.} \qquad (5.27)$$

Eq. (5.27) can be simplified further by using the regulator eq. (4.87), p. 133 with

$$R_k(p^2) = (k^2-p^2)\,\Theta(k^2-p^2) \qquad (5.28)$$
$$\dot{R}_k(p^2) = 2k^2\,\Theta(k^2-p^2)$$

with $\dot{R}_k(p^2)\dfrac{1}{p^2+R_k(p^2)+\omega} = 2\,\dfrac{1}{1+\omega/k^2}\,\Theta(k^2-p^2)$ (5.29)

leading to

$$\boxed{\partial_t V_k(\rho) = \frac{k^d}{d}\frac{\Omega_d}{(2\pi)^d}\left\{ \frac{N-1}{1+V'(\rho)/k^2} + \frac{1}{1+\frac{V'(\rho)+2\rho V''(\rho)}{k^2}} \right.} \qquad (5.30)$$

with

$$\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$$

As in the Ising model we introduce dim. less variables, see eq. (4.93), p. 135:

$$\boxed{\begin{aligned} \hat{V}(\hat{\rho}) &= \frac{1}{k^d} V(\rho) \\ \hat{\rho} &= \rho\, k^{2-d} \end{aligned}} \qquad (5.31)$$

This leads to

$$\boxed{\begin{aligned} &\left(\partial_t\big|_{\hat{\rho}} + d + (2-d)\,\hat{\rho}\,\partial_{\hat{\rho}}\right)\hat{V} \\ &\quad = \frac{1}{d}\,\frac{\Omega_d}{(2\pi)^d}\left\{\frac{N-1}{1+\hat{V}'} + \frac{1}{1+\hat{V}'+2\hat{\rho}\hat{V}''}\right\} \end{aligned}} \qquad (5.32)$$

With eq. (5.32) we perform the fixed point analysis analoguously to that done in chapter 4.3, eqs. (4.94), (4.95), ff. For $N=1$, the Ising case, eq. (5.32) and eq. (4.94) agree.

The FP equation reads, $\dot{\hat{v}} = 0$:

$$\boxed{\left(d + (2-d)\,\hat{\rho}\partial_{\hat{\rho}}\right)\hat{V} = \frac{1}{d}\frac{\Omega_d}{(2\pi)^d}\left\{\frac{N-1}{1+\hat{V}'} + \frac{1}{1+\hat{V}'+\hat{\rho}\hat{V}''}\right\}} \tag{5.33}$$

Eq. (5.33) can be solved numerically in a Polynomial expansion, again see the webpage of the lecture (p. 136) for a mathematica program; or on a grid.

Here we again resort to the simplest approximation

$$V = \lambda/2\,(\rho-\kappa)^2 \tag{5.34a}$$

with

$$\begin{aligned} \partial_t V(\rho) &= \frac{1}{2}\,\dot{\lambda}(\rho-\kappa)^2 - \lambda\dot{\kappa}(\rho-\kappa), \\ V'+2\rho V'' &= \lambda(\rho-\kappa) + 2\lambda\rho \\ V' &= \lambda(\rho-\kappa) \end{aligned} \tag{5.34b}$$

where we have relabeled $\hat{\rho}\to\rho$, $\hat{V}\to V$, ..., see also p. 136.

Following the route in the $N=1$ case, we have for $\kappa>0$:

$$\partial_\rho^2\Big|_{\rho=\kappa}\left(\partial_t + d + (2-d)\rho\partial_\rho\right)V\Big|_{\rho=\kappa} = \beta_\lambda - (4-d)\lambda$$

$$\partial_\rho\Big|_{\rho=\kappa}\left(\qquad\right)V\Big|_{\rho=\kappa} = -\lambda\,\beta_\kappa - (2-d)\kappa\lambda \tag{5.35}$$

and also:

$$\partial_\rho^2\Big|_{\rho=\kappa}\big(\underbrace{\mathrm{Flow}_{FP}}_{\text{rhs of eq.(5.33)}}\big)\Big|_{\rho=\kappa} = \frac{2}{d}\frac{\Omega_d}{(2\pi)^d}\lambda^2\left[\underbrace{(N-1)}_{\text{Goldstones}} + \frac{9}{(1+2\lambda\kappa)^3}\right]$$

$$\partial_\rho\Big|_{\rho=\kappa}\left(\mathrm{Flow}_{FP}\right)\Big|_{\rho=\kappa} = -\frac{1}{d}\frac{\Omega_d}{(2\pi)^d}\lambda\left[(N-1) + \frac{3}{(1+2\lambda\kappa)^2}\right] \tag{5.36}$$

leading to the fixed point equation

$$\boxed{\begin{aligned}\beta_\lambda &= (d-4)\lambda + \frac{2}{d}\frac{\Omega_d}{(2\pi)^d}\lambda^2\left[(N-1) + \frac{9}{(1+2\lambda\kappa)^3}\right]\\ \beta_\kappa &= (2-d)\kappa + \frac{1}{d}\frac{\Omega_d}{(2\pi)^d}\lambda\left[(N-1) + \frac{3}{(1+2\lambda\kappa)^2}\right]\end{aligned}} \tag{5.37}$$

In 3d this leads to (using eq. (4.101), p. 138,

$$\boxed{\begin{aligned} \beta_\lambda &= -\lambda + \frac{4}{3\pi^2}\lambda^2\left[(N-1) + \frac{9}{(1+2\lambda\kappa)^3}\right] \\ \beta_\kappa &= -\kappa + \frac{1}{6\pi^2}\lambda\left[(N-1) + \frac{3}{(1+2\lambda\kappa)^2}\right] \end{aligned}} \quad (5.38)$$

leading to the stability matrix

$$B_{ij}(N) = B_{ij}(1) + (N-1)\begin{pmatrix} 0 & 0 \\ \frac{1}{6\pi^2} & \frac{2}{3\pi^2}\lambda \end{pmatrix} \quad (5.39)$$

The eigenvalues ω_0, ω_1 are given $\boxed{N=2}$,

$$\omega_0 = -1.611 \quad , \quad \omega_1 = 0.384 \qquad (5.40)$$

and hence we have

$$\boxed{\nu = 0.621 \qquad \omega_1 = 0.384} \quad (5.41)$$

$N=2$

$\nu \approx 0.67$ (theoretical)

see Pelissetto, Vicari, Phys. Rep., p. 67

(5.42)

Result for ϕ^{12}:

$$\nu = 0.71 \tag{5.43}$$

and for $\phi^M, M \to \infty$: $\boxed{\nu = 0.71.}$

see Litim '02, p. 10.

This is still not fully conclusive for the distinction of $N=1$ and $N=2$.

Adding the next correction,

$$Z_k (\partial_\nu \phi)^2 \tag{5.44}$$

to the approximation of the effective action adds the second independent critical exponent η, to wit (ϕ^{12})

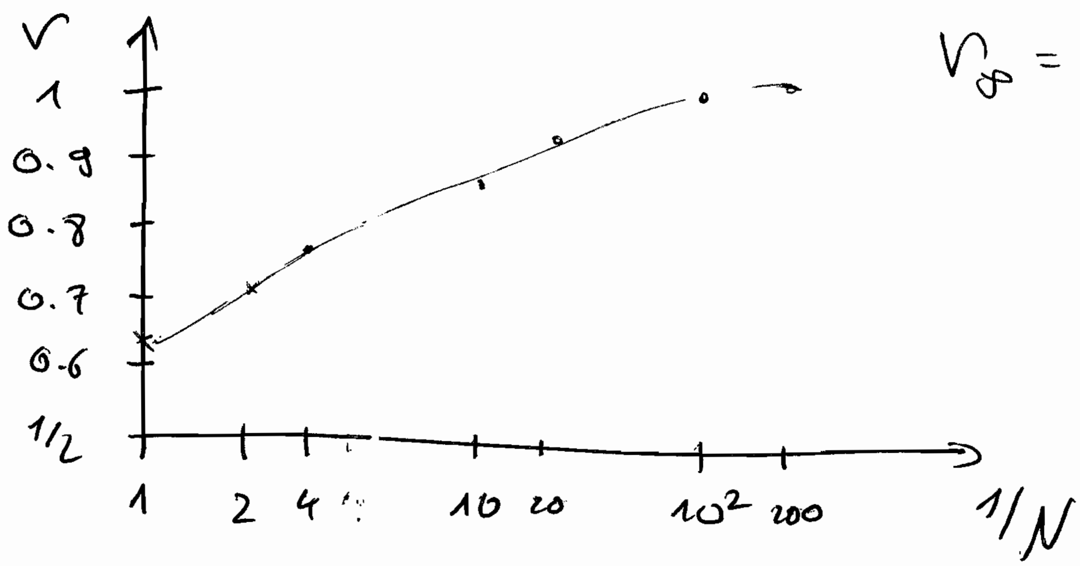
$$\begin{array}{cc} \nu = 0.64, & \nu = 0.69 \\ N=1 & N=2 \\ \eta = 0.044 & \eta = 0.044 \\ \eta_{theor} \simeq 0.036 & \eta_{theo} \simeq 0.038 \end{array} \tag{5.45}$$

Let me also mention the state-of-the-art FRG results (order ∂^4): (Ising)

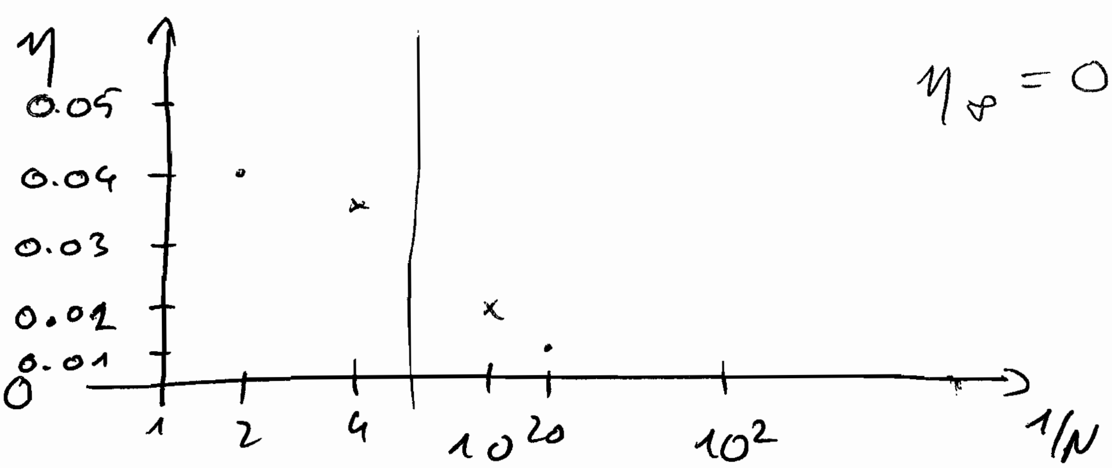
$$\nu = 0.63 \quad , \eta = 0.036 \tag{5.46}$$

Canet et al '03
Litim, Zappala '10

We conclude our numerical analysis with studying the N-dependence of the critical exponents,



$\nu_\infty = 1$



$\eta_\infty = 0$

We close this section with two analytical applications of the flow equation

(1) Mermin-Wagner theorem

(2) Large N limit : $N \to \infty$

(1) The Mermin-Wagner theorem states that in dimensions $d \leq 2$ there is no sponto symmetry breaking of a continuous symmetry.

We recall the flow eq. (5.30) for the effective potential

$$\partial_t V_k = \frac{k^d}{d} \frac{\Omega_d}{(2\pi)^d} \left\{ \frac{N-1}{1+V'/k^2} + \frac{1}{1+\frac{V'+2\rho V''}{k^2}} \right\} \quad (5.47)$$

for a theory with discrete symmetry, $N=1$, and continuous symmetries, $N > 1$.

Remember, that the first term is generated by the Goldstone modes with

$$V_k'(\rho_{\min k}) = 0 \tag{5.48}$$

for $\rho_{\min k} > 0$ in the broken phase.

The Mermin-Wagner theorem states that

$$\boxed{\rho_{\min k=0} = 0,} \tag{5.49}$$

the theory is in the symmetric phase for $k=0$, that is $L \simeq 1/k = \infty$, where L is the size of the system.

Now we assume that $\rho_{\min \Lambda} > 0$ at some $\Lambda \neq 0$. The flow of the minimum can be determined by taking the $t = \ln k$-derivative of eq. (5.48).

$$\rho_{\min\, k=0} > 0, \tag{5.53}$$

that is spontaneous symmetry breaking.

(b) For $d \leq 2$ the flow stays finite in the limit $k \to 0$ $(t \to -\infty)$ or even diverges, $\kappa < 2$. We conclude that

$$\boxed{\rho_{\min\, k=0} = 0} \tag{5.54}$$

for arbitrary initial conditions: Note that

$$\lim_{k \to 0} \int_{t_0}^{\ln k} dt' \, \dot{\rho}_{\min\, k} \simeq \lim_{k \to 0} \int_{t_0}^{\ln k} \frac{dk'}{k'} \, k'^{d-2} \frac{\Omega_d}{(2\pi)^d} (N-1) \to \infty \tag{5.55}$$

It is important to note that $\hat{\rho}_{\min}$ might still be finite. This is important for the discussion of the Berezinsky - Kosterlitz - Thouless phase transition (vortices).

This leads to

$$\partial_t \left[V_k'(\rho_{\min k}) \right] = 0 = \dot{\rho}_{\min k} V_k''(\rho_{\min k}) + \dot{V}_k'(\rho_{\min k}) \tag{5.50}$$

and hence

$$\dot{\rho}_{\min k} = - \frac{\dot{V}_k'(\rho_{\min k})}{V_k''(\rho_{\min k})} \tag{5.51}$$

With $0 < V''(\rho_{0k}) < \infty$ we are left with

$$\dot{\rho}_{\min k} = + \frac{k^{d-2}}{d} \frac{\Omega d}{(2\pi)^d} \left\{ (N-1) + \frac{3 + 2\rho_{\min} V'''/V''}{\left(1 + \underbrace{\frac{2\rho_{\min} V''}{k^2}}_{V_k'(\rho_0)=0}\right)^2} \right\} [\rho_0] \tag{5.52}$$

In the limit $k \to 0$ the second term on the rhs of eq. (5.52) vanishes with k^{d+2}. The first term vanishes with k^{d-2}!

(a) For $d > 2$ this entails that the flow of $\dot{\rho}_{\min}$ decays with k^{d-2} at small k. Subject to the initial $\rho_{\min \Lambda}$ this can lead to

The above analysis was done in a specific approximation which, however, carries the essence of the underlying physics: in lower dimensions quantum fluctuations get successively more important. This is encoded in the relative weight

$$\frac{N-1}{1+V'/k^2} \bigg/ \frac{1}{1+\frac{V'+2\rho V''}{k^2}} \rightarrow \frac{k^2}{2\rho V''}(N-1) \qquad (5.56)$$

$\sim k^2$ which survives also in the full case.

(b) The relative weight is increased in the limit $N \to \infty$. Indeed this limit invalidates the above Mermin-Wagner argum. Moreover it allows a semi-analytic solution of eq. (5.47).

To that end we rescale V and ρ:

$$V \to V/(N-1), \qquad (5.57)$$

$$\rho \to \rho/N-1,$$

and eq. (5.47) turns into

$$\partial_t V_k = \frac{k^d}{d} \frac{\Omega_d}{(2\pi)^d} \left\{ \frac{1}{1+V'/k^2} + \frac{1}{(N-1)} \frac{1}{1+\frac{V'+2\rho V''}{k^2}} \right\} \qquad (5.58)$$

Again we make use of the dim. less variables, eq. (5.31), and arrive at, $u = \hat{V}'$, $N \to \infty$,

$$\boxed{\mathring{u} + (2-d)\hat{\rho}\, u' + A_d \frac{u'}{(1+u)^2} + 2u = 0} \qquad (5.59)$$

with $A_d = \frac{1}{d} \frac{\Omega_d}{(2\pi)^d}$. With the method of characteristics this is equivalent to solving

$$\frac{dt}{ds} = 1 \;, \quad \frac{d\hat{\rho}}{ds} = (d-2)\hat{\rho} + A_d/(1+w)^2 \;, \quad \frac{dw}{ds} = -2w \qquad (5.60)$$

for $t(s)$, $\hat{\rho}(s)$, $w(s)$, and

$$w := u(t, \rho)$$

It follows that $\boxed{t = s = \ln 1/k}$, and hence

$$\boxed{w(s) = w(0)\cdot e^{-2s}} \qquad (5.61a)$$

Inserting this into $d\hat\rho/ds$ or rather $d\rho/ds$ leads to

$$\boxed{\rho = \rho_0(\rho) + A_d \int_0^t ds \, \frac{e^{(d-2)s}}{\left(1 + u_0(\hat\rho_0(\hat\rho))\, e^{-2s}\right)^2}} \qquad (5.61b)$$

fct. of t, s respectively

where

$$u_0(\hat\rho_0) = u_{t=0}(\hat\rho_0)$$

in a slight abuse of notation ($t=0 \Leftrightarrow k=1$).

One can easily prove by insertion that $u_t(\hat\rho)$ defined by eq. (5.61) solves eq. (5.58).

Even though eq. (5.61) still looks complicated, it has far-reaching consequences and allows for a simple discussion of the qualitative features of the solution. Moreover, the implicit eq. (5.61b) is simple to solve numerically

Remarks:

(i) The solution entails that the ρ-deriv. of the potential takes the same values as the derivat. of the input pot. (classical part, but evaluated at $\rho_t(\rho)$.

(ii) $\rho_t(\rho)$ is always bigger than ρ, for $k<\Lambda$! The integrand in eq. (5.61b) is bigger than zero and $t<0$. This entails that the flow has symmetry-restoring properties, or, put differently, quantum fluctuations smoothen the phase transition

(iii) The question after the existence of a phase transition can be answered with the flow of ρ_{min}.

This is given analytically: First we notice that

$$\rho_{min} = \rho\,(\rho_{0,min}) \tag{5.62}$$

Hence

$$\begin{aligned}\rho_{min} &= \rho_{0\,min} + A_d \int_0^t ds\; e^{(d-2)s} \\ &= \rho_{0\,min} - \frac{A_d}{(d-2)}\left(1 - e^{(d-2)t}\right)\end{aligned} \tag{5.63}$$

and, at $k=0$,

$$\boxed{\rho_{min} = \rho_{0\,min} - \frac{A_d}{d-2}}$$

Hence, at $d=2$, $\rho_{min} = 0$.

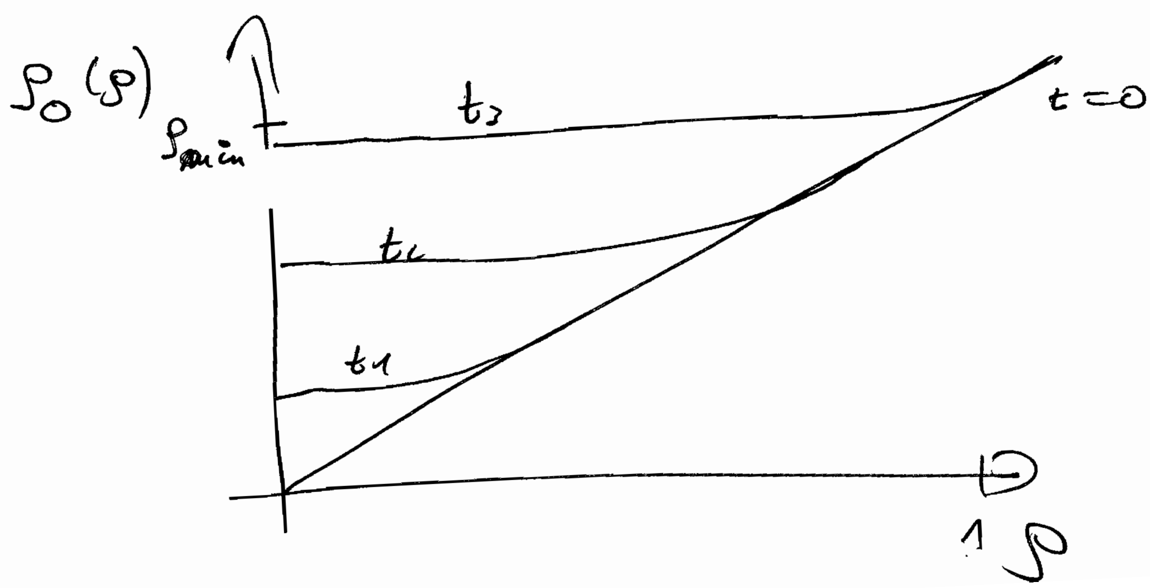


Fig 5.1

It follows that $V_k(\rho)$ can be read-off from the initial potential at $k=1$:

$$u_0(\hat{\rho}_0) = \hat{V}_1^{\,1}(\hat{\rho}_0)\, e^{-2t} \qquad (5.64)$$

with

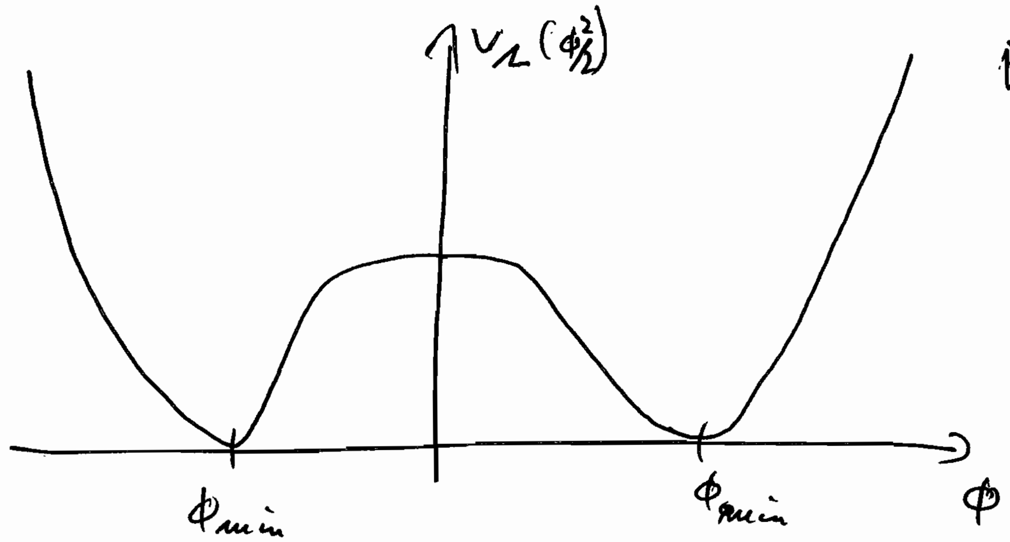


Fig 5.2

and Fig 5.1. We are led to