Critical Phenomena

1) One dimensional physical systems, in which the interactions have a finite length, can always be divided into subsystems $\Gamma_i, i = 1..N$ such that there is interaction only between neighbouring susbsystems, such that the Hamiltonian is written as:

$$H(\mathbf{S}) = \sum_i H_0(s_i) + \sum_i H'(s_i, s_{i+1}) \tag{1}$$

where s_i is the classical variable that describes the i-th subsystem. When s_i can only take values in a finite set, we write the *transfer matrix* of the system as:

$$\langle s_i|\hat{T}|s_{i+1}\rangle = \exp\left[-\beta\left(\frac{1}{2}H_0(s_i) + \frac{1}{2}H_0(s_{i+1}) + H'(s_i, s_{i+1})\right)\right] \tag{2}$$

We assume that the system is periodic, i.e. $\Gamma_{N+1} \equiv \Gamma_1$.

a) Show that

$$Z = \mathrm{tr}(\hat{T}^N) \tag{3}$$

b) Show that in the limit $N \to \infty$ the free energy can be written as:

$$F = -NkT \ln \lambda_1 \tag{4}$$

where λ_1 is the biggest eigenvalue of the transfer matrix.

2) The Ising model is defined by the Hamiltonian:

$$H = -J\sum_{i=1}^{N} s_i s_{i+1} \tag{5}$$

Use the transfer matrix method to calculate the following quantities:

a) the partition function,

b) the internal energy and specific heat of the system,

c) the correlation function $\langle s_i s_j\rangle$, and its fourier transform:

$$G(q) = \sum_{n=-\infty}^{n=\infty} e^{iqn}\langle s_0 s_n\rangle \tag{6}$$

d) the susceptibility χ, and the correlation length ξ using the result for the correlation function calculated above.