1. Symmetries & topological excitations

1.1. Complex scalar field in d dimensions

Lagrangian

$$-\tfrac{1}{2}\left(\partial_\nu\phi_1\partial^\nu\phi + \partial_\nu\phi_2\partial^\nu\phi\right)$$
$$\shortparallel$$
$$\mathcal{L} = -(\partial_\nu\phi^*)(\partial^\nu\phi) - V(\phi^*\phi) \qquad (1.1)$$

with $\quad \phi = 1/\sqrt{2}\,(\phi_1 + i\phi_2)$

EoM: $\quad \mathcal{L} = \mathcal{L}(\phi, \phi^*; \partial_\nu\phi, \partial_\nu\phi^*)$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} = 0 \xrightarrow{(*)} \frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi^*)} = 0 \qquad (1.2)$$

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from action principle

$$S[\phi] = \int d^dx\, \mathcal{L}(\phi) \qquad (1.3)$$

$$\frac{\delta S}{\delta\phi} = 0 \Rightarrow \text{EoM} \quad \text{with} \quad \frac{\delta\phi(x)}{\delta\phi(y)} = \delta(x-y)$$

see p. 1a

$$= \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi} = \partial_\nu\partial^\nu\phi^* - \frac{\partial V}{\partial\phi}$$

$$\frac{\delta S}{\delta \phi(y)} = 0 = \int d^d x \; \frac{\delta \mathcal{L}(\phi, \partial\phi)}{\delta \phi(y)}$$

$$= \int d^d x \left(\frac{\delta \phi(x)}{\delta \phi(y)} \frac{\partial \mathcal{L}}{\partial \phi}(x) + \frac{\delta\, \partial_\nu \phi(x)}{\delta \phi(y)} \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi}(x) \right)$$

$$= \int d^d x \left(\delta(x-y) \frac{\partial \mathcal{L}}{\partial \phi}(x) + \partial_\nu^x \delta(x-y) \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi}(x) \right)$$

$$= \frac{\partial \mathcal{L}}{\partial \phi}(y) - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi}(y)$$

(1.4)

free field: $V = m^2 \phi^* \phi = \frac{1}{2} m^2 \left(\phi_1^2 + \phi_2^2\right)$

$$\frac{\delta S}{\delta \phi} = 0 \quad : \quad \left(\partial_\nu \partial^\nu - m^2\right) \phi(x) = 0 \qquad (1.5)$$

Fourier trafo: $\phi(x) = \int \frac{d^d p}{(2\pi)^d} \tilde{\phi}(p)\, e^{i p_\nu x^\nu}$

$$\Rightarrow \quad \underbrace{\left(p^2 + m^2\right)}_{p^2 = p_\nu p^\nu} \tilde{\phi}(p) = 0 \qquad (1.6)$$

$\Rightarrow$ wave pakages $\tilde{\phi}(p)$ with $p^2 + m^2 = 0$

$$\tilde{\phi} = \delta\left(p^2 - m^2\right) a\left(\vec{p}, p_0\right) \qquad (1.7)$$

Symmetries: $U(1)$

$$\phi(x) \to e^{i\omega} \phi(x) \xrightarrow{*} \phi^*(x) \to \phi^*(x) e^{-i\omega} \qquad (1.8)$$

with constant ω: $\boxed{\partial_\nu \omega = 0}$

global symmetry

infinitesimal: $\phi(x) \to (1+i\omega(x))\,\phi(x)$

+ local $\phi^*(x) \to \phi^*(x)\,(1-i\omega(x))$ (1.9)

action:

$$S[e^{i\alpha}\phi] = S[(1+i\omega)\phi] + \mathcal{O}(\omega^2)$$

$$= S[\phi] + i\int d^dx\,\omega(x)\,\frac{\delta S}{\delta\alpha(x)} + \mathcal{O}(\omega^2)$$

EoM: $\left.\frac{\delta S}{\delta i\omega(x)}\right|_{\alpha=0} = 0$ from $\delta\phi = i\omega\phi$

$$\Rightarrow \underbrace{\frac{\partial\mathcal{L}}{\partial\omega}}_{\text{global sym.: } 0} - \partial_\mu \left.\frac{\partial\mathcal{L}}{\partial\partial_\mu\omega}\right|_{EoM} = 0 \qquad (1.10)$$

$$\Rightarrow \boxed{\partial_\nu \left.\frac{\partial\mathcal{L}}{\partial\partial_\nu\omega}\right|_{EoM} = 0} \qquad (1.11)$$

$$\Rightarrow \partial_\nu J^\nu = 0 \quad \text{with } J^\nu = \frac{\partial\mathcal{L}}{\partial\partial_\nu\omega} \qquad (1.12)$$

Noether current J^ν:

$$\partial_\nu(1+i\omega)\phi = i\partial_\nu\omega\,\phi + (1+i\omega)\,\partial_\nu\phi \qquad (1.13)$$

with $\frac{\partial\,\partial_\nu(1+i\omega)\phi}{\partial\partial_\nu\omega} = i\phi$

$$\Rightarrow \boxed{J^\nu = i\left(\phi \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} - \phi^* \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi^*}\right)} \qquad (1.14)$$

Noether charge:

$$Q(t) = \int d^{d-1}x \, J^0$$

$$= i\int d^{d-1}x \left(\phi \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} - \phi^* \frac{\partial \mathcal{L}}{\partial \partial_0 \phi^*}\right)$$

$$\Rightarrow \boxed{Q(t) = -i \int d^{d-1}x \left(\phi\, \partial^0 \phi^* - \phi^* \partial^0 \phi\right)} \qquad (1.15)$$

Q conserved charge: (on EoM)

$$\dot{Q} = \partial_0 Q = -i\int d^{d-1}x \left(\phi\, \partial_0\partial^0 \phi^* - \phi^* \partial_0 \partial^0 \phi\right)$$

EoM: $\partial_0 \partial^0 \phi = \vec{\partial}^2 \phi + \frac{\partial V}{\partial \phi^*}$ $\rightarrow$

$$= -i\int d^{d-1}x \left(\phi\, \vec{\partial}^2 \phi^* - \phi^* \vec{\partial}^2 \phi\right)$$

$$= -i\int d^{d-1}x \, \vec{\partial}\left(\phi\, \vec{\partial} \phi^* - \phi^* \vec{\partial} \phi\right)$$

$$= 0 \qquad (1.16)$$

for sufficiently fast decaying fields: next page

Introduce manifold $\mathcal{M}$:

$$-i \int_{\mathcal{M}} d^{d-1}x\, \vec{\partial}\left(\phi\, \vec{\partial}\, \phi^* - \phi^*\, \vec{\partial}\, \phi\right)$$

$$= -i \int_{\partial\mathcal{M}=F} d\vec{F} \cdot \left(\phi\, \vec{\partial}\, \phi^* - \phi^*\, \vec{\partial}\, \phi\right) \overset{!}{=} 0 \qquad (1.17)$$

Noether theorem: 'Continuous global symmetries have locally conserved currents $\partial_\nu J^\nu = 0$ charges $\dot{Q} = 0$ on the EoM (dynamics) subject to appropriate boundary conditions.'

1st lecture

Comments (2nd lecture)

(1) home page lecture
script

(2) 'Kummerkasten's

https://mathphys.fsk.uni-heidelberg.de/kummerkasten

(3) Mathematics

- Homotopy
- Homology + Co-hom.
- (Holonomy)
- Index theorems
- fibre bundles

Noether theorem in classical scalar theory

charge, current $j^\mu = \left[\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - (\ )^*\right] i$

$$V[\phi] = V[\phi \phi^*]$$

discrete Symmetries:

Real scalar field:

$$\mathcal{L} = -\frac{1}{2}\partial_\nu\phi\,\partial^\nu\phi - V(\phi) \qquad 1.18$$

EoM:

$$-\partial_\nu\partial^\nu\phi + \frac{\partial V}{\partial\phi} = 0 \qquad (1.19)$$

with $(\phi) \geq 0$

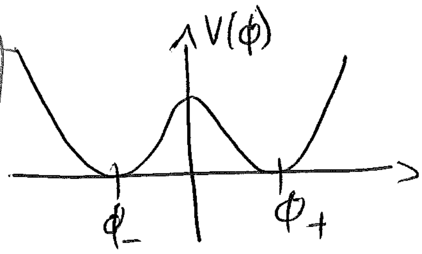
Examples: $d = 1+1$

(i) ϕ^4 - theory

$$V_{\phi^4}(\phi) = \lambda/4!\left(\phi^2 - \frac{6\mu^2}{\lambda}\right)^2 \qquad (1.20)$$

$$= -\frac{1}{2}\mu^2\phi^2 + \lambda/4!\,\phi^4 + 3\mu^4/2\lambda$$

minima: $\boxed{\phi_\pm = \pm\sqrt{6\mu^2/\lambda}}$



EoM: $\boxed{\partial_\nu\partial^\nu\phi = -\mu^2\phi + \lambda/6\,\phi^3}$ (1.21)

Symmetry: $\phi \to -\phi$

(ii) sine-Gordon model

$$V_{SG}(\phi) = \mu^2/\lambda^2 \,(1 - \cos \lambda \phi) \qquad (1.22)$$

$\phi \ll 1/\lambda$: $\simeq -\mu^2/2\, \phi^2 + \mu^2 \lambda^2/4!\, \phi^4 + \mathcal{O}(\phi^6)$

periodic with Minima

$$\boxed{\phi_n = 2\pi/\lambda\, n} \quad \text{with } n \in \mathbb{Z} \qquad (1.23)$$

EoM :

$$\boxed{\partial_\nu \partial^\nu \phi = \mu^2/\lambda\, \sin \lambda \phi} \qquad (1.24)$$

Symmetry: $\phi \to \phi + \frac{2\pi}{\lambda} n, \quad n \in \mathbb{Z}$

Physics for config. with $\boxed{\text{finite}}$ Energy:

$$H = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\partial_1 \phi)^2 + V(\phi) \right] \quad (1.25)$$

$$H < \infty : \; |x|(\partial_\mu \phi)^2 ,\; |x| V(\varphi) \to 0 \qquad \text{for } |x| \to \infty$$

$$\Rightarrow \quad V(\phi_{\pm\infty}) = 0 \qquad \phi_{\pm\infty} = \phi(x \to \pm\infty)$$

For example:

$$\phi(x,t) = \phi_0 \qquad \phi_0 = \begin{cases} \pm\sqrt{6\mu^2/\lambda} \\ 2\pi/\lambda \, n \end{cases}$$

(1.26)

Exitations:

$$\phi(x,t) = \phi_0 + \varphi(x,t)$$

$$\Rightarrow \mathcal{L}(\phi; \partial\phi) = \underbrace{\mathcal{L}(\phi_0)}_{0} + \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right)_{\phi = \phi_0}}_{\text{EoM} \to 0} \varphi$$

$$\frac{1}{2}\left[-(\partial_0^2 + \partial_1^2) + \frac{\partial^2 V}{\partial \phi^2}(\phi_0) \right] \varphi^2 + \mathcal{O}(\varphi^3)$$

(1.27)

$\Rightarrow$ small fluctuations propagate as

free 'particles/waves" with mass $\frac{\partial^2 V}{\partial \varphi^2}(\phi_0)$

$$\frac{\partial^2 V}{\partial \phi^2}(\phi_0) = \begin{cases} 2\mu^2 & \phi^4\text{-theory} \\ \mu^2 & \text{sine-Gordon} \end{cases} \tag{1.28}$$

Gaußian approximation/saddle-point approx.

Question: e.g. ϕ^4-theory

'Is there a config. ϕ with $H(\phi) < \infty$

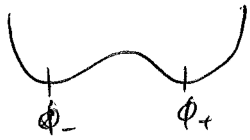
and
$$\phi(x, t \to -\infty) = \phi_- \tag{1.29}$$
$$\phi(x, t \to +\infty) = \phi_+$$

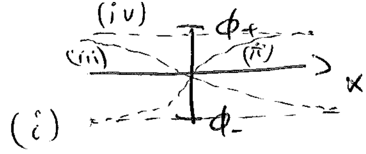
Answer: $\boxed{\text{No}}$: $\int_{-\infty}^{+\infty} dx V(\phi \neq \phi_\pm) = \infty$ $\Rightarrow$ no smooth config ϕ with (1.29)

$$\Rightarrow \boxed{\partial_t \phi(x \to \pm\infty, t) = 0} \quad \text{'conserved'} \tag{1.30}$$

but $\phi(+\infty, t) \neq \phi(-\infty, t)$ possible!

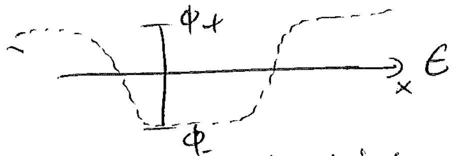
4 possibilities:





		ϕ_0	Energy	sectors
(i)	$\phi_{-\infty} = \phi_-$, $\phi_{+\infty} = \phi_-$	ϕ_{--}	E_{--}	vacuum
(ii)	$\phi_{-\infty} = \phi_-$, $\phi_{+\infty} = \phi_+$	ϕ_{-+}	E_{-+}	kink
(iii)	$\phi_{-\infty} = \phi_+$, $\phi_{+\infty} = \phi_-$	ϕ_{+-}	E_{+-}	kink
(iv)	$\phi_{-\infty} = \phi_+$, $\phi_{+\infty} = \phi_+$	ϕ_{++}	E_{++}	vacuum

because



kink – anti-kink $\sim \phi_{++}$

In summary:

$$\phi_\lambda = \phi_0 + \lambda \cdot \underset{\text{fluctuation}}{\varphi} \qquad \lambda \in \mathbb{R} \text{ general} \qquad (1.31)$$

with $\boxed{H(\phi_\lambda) < \infty}$: $\boxed{\lim_{x \to \pm\infty} |x|\,\varphi(x) \to 0}$

$$\Rightarrow \boxed{\phi_\lambda(x \to \pm\infty, t) = \phi_0} \qquad \text{Conserved} \quad (1.32)$$

$\rightsquigarrow$ disconnected sectors characterised by $\phi(x \to \pm\infty, t)$

discrete Symmetry: $\phi \to -\phi$

$$E_{--} \to E_{++} \qquad \left(H(-\phi) = H(\phi) \right)$$
$$E_{+-} \to E_{-+} \qquad (1.33)$$

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$$Q_{top} = \frac{1}{2}\sqrt{\frac{\lambda}{6\mu^2}}\left(\phi_{+\infty} - \phi_{-\infty}\right) \quad (\text{mod } 2)$$

$\uparrow$

symmetry

$\downarrow$

$$= \frac{1}{2}\sqrt{\frac{\lambda}{6\mu^2}}\int dx\, \frac{\partial \phi}{\partial x} \quad (\text{mod } 2)$$

$$Q_{top} \in \{0, 1, -1\}$$

$$\Big\downarrow \text{ mod } 2$$

$$\mathbb{Z}_2$$

Sine - Gordon:

Symmetry: $\phi \to \phi + {}^{2\pi}/_{\lambda}\, l \quad , \quad l \in \mathbb{Z}$

sectors: (n, m)

$$\phi_{-\infty} = {}^{2\pi}/_{\lambda}\, n \quad , \quad \phi_{+\infty} = {}^{2\pi}/_{\lambda}\, m \quad , \quad E_{n,m} \tag{1.34}$$

Symmetry: $\phi_{n,m} \to \phi_{n+l,\, m+l} \sim \phi_{n,m}$

$$E_{n,m} \to E_{n+l,\, m+l} \tag{1.35}$$

$\Rightarrow \quad Q_{top} = m - n \qquad$ characterised sectors

sym.: $\quad Q_{top} \to m+l-(n+l) = m-n$

$$Q_{top} = \frac{\lambda}{2\pi}\left(\phi_{+\infty} - \phi_{-\infty}\right)$$

$$= \frac{\lambda}{2\pi}\int_{-\infty}^{+\infty} dx \, \frac{\partial \phi}{\partial x} \tag{1.36}$$

- no dynamical information (pure boundary term)

Current: we want
(relativistic)

$$\partial_\mu J^\mu_{top} = 0 \qquad (1.37)$$

and $$Q_{top} = \int_{\mathbb{R}} dx \, J^0_{top}$$

$$\Rightarrow \boxed{J^\mu_{top} \simeq \varepsilon^{\mu\nu} \partial_\nu \phi} \qquad (1.38)$$

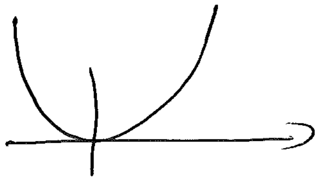
$$\rightsquigarrow \partial_\mu J^\mu_{top} = \partial_\mu \varepsilon^{\mu\nu} \partial_\nu \phi = \tfrac{1}{2} [\partial_\mu, \partial_\nu] \varepsilon^{\mu\nu} \phi$$

$$\xrightarrow{\text{integrability}} \overset{!}{=} 0 \qquad (1.39)$$

Comments:

(1) Conservation of top. charge Q_{top} follows from finite energy constraint

(2) Q_{top} well-defined for all scalar field theory in 1+1 dimensions

(a) $V(\phi) =$ 

$\Rightarrow \lim_{x \to \pm\infty} \phi(x,t) = 0$ for sol. of EoM

$\rightsquigarrow Q_{top} \simeq (\phi_{+\infty} - \phi_{-\infty}) = 0$ (1.40)

(b) $V(\phi) = 0$ free theory $\mathcal{L} = -\frac{1}{2}\partial_\nu\phi\,\partial^\nu\phi$

const $\phi \in \mathbb{R}$ sol. of EoM

$\Rightarrow Q_{top} \simeq (\phi_{+\infty} - \phi_{-\infty}) \in \mathbb{R}$ (1.41)

$\Rightarrow U(\phi)$ determines vacuum manifold Ω

(c) double well $\Omega = \{\phi_\pm\}$

(d) sin Gordon $\Omega \cong \mathbb{Z}$

(3) Top. charge already quantised in classical field theory;
in 1+1 scalar theories: classification of maps

$$\partial\mathbb{R} \longrightarrow \Omega$$
$$\{\pm\infty\} \longrightarrow \Omega$$

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(a) $\Omega = \{0\}$

$\{\pm\infty\} \to \{0\}$ $\quad Q_{top} = 0$

(b) $\Omega = \mathbb{R}$

$\{\pm\infty\} \to \mathbb{R}$ $\quad Q_{top} = \mathbb{R}$

(c) $\Omega = \{\Phi_-, \Phi_+\}$

$\{-\infty, +\infty\} \to \{\Phi_-, \Phi_+\}$ $\quad Q_{top} = \{0, 1, -1\}$

(d) $\Omega = \{\Phi_{n,m}\}$

$\{\pm\infty\} \to \{\Phi_{n,m}\}$ $\quad Q_{top} = \mathbb{Z}$