1.2 Solitons

static non-trivial solutions to EoM

We search for solutions to the EoM

in a given sector (with Q_{top})

Static energy: (p.7)

$$H_{static}[\phi] = \int \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + V(\phi) \right] dx \tag{1.42}$$

static EoM (sEoM):

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{\phi_0} = \left.\frac{\partial V}{\partial \phi}\right|_{\phi_0} \tag{1.43}$$

$\phi_0(x)$ satisfies full EoM with $\phi_0(x,t) = \phi_0(x)$

(1) classical vacuum ϕ_0 const. with $V'(\phi_0)=0$ and $V(\phi_0)$ minimum $(=0)$

$$H_{static}[\phi] \simeq 0 \tag{1.44}$$

(2) non-trivial solutions

consider (gEoM) $\cdot \frac{\partial \phi_0}{\partial x}$
(1.43)

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial \phi_0}{\partial x}\right)^2\right] = \frac{\partial}{\partial x} V(\phi_0) \qquad (1.45)$$

$$\rightsquigarrow \left(\frac{\partial \phi_0}{\partial x}\right)^2 = 2\,V(\phi_0) + C \qquad (1.46)$$

However $\left.\frac{\partial \phi_0}{\partial x}\right|_{x\to\pm\infty} = 0 = \left. V(\phi_0)\right|_{x\to\pm\infty} \Rightarrow C = 0$

for finite H

$$\Rightarrow \boxed{\left(\frac{\partial \phi_0}{\partial x}\right)^2 = 2\,V(\phi_0)} \qquad (1.47)$$

$V \geq 0$:

$$\boxed{\frac{\partial \phi_0}{\partial x} = \pm\sqrt{2V(\phi_0)}} \qquad (1.48)$$

Repeat:

Kinks on p 16

remind: only neighbouring kinks

$$(1.49) \quad x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{2V(\phi)}}$$

with $\boxed{V(\phi_{0n}) = 0}$

Examples: p. 16

solutions

(1) ϕ_0 const $\wedge$ $V(\phi_0)=0$ classical vacuum

(2) non-trivial, monotonic solution

$$x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{2V(\phi)}} \qquad (1.49)$$

interpolate between <u>neighbouring</u> vacua

Remark: Note that $V(\phi_{0n}) = 0$ $\quad \phi_{0n}$: minima of V with $V(\phi_0)=0$

$\Rightarrow |x - x_0| = \infty$ if $\phi(x), \phi(x_0) = \phi_{0n}$

$\Rightarrow$ only interpolation between neighbouring vacua possible (strictly speaking)

Examples:

(a) ϕ^4-Kink: $\boxed{\phi_0(x) = \sqrt{6\mu^2/\lambda}\,\tanh\left[\frac{\mu}{\sqrt{2}}(x-x_0)\right]}$
p. 6

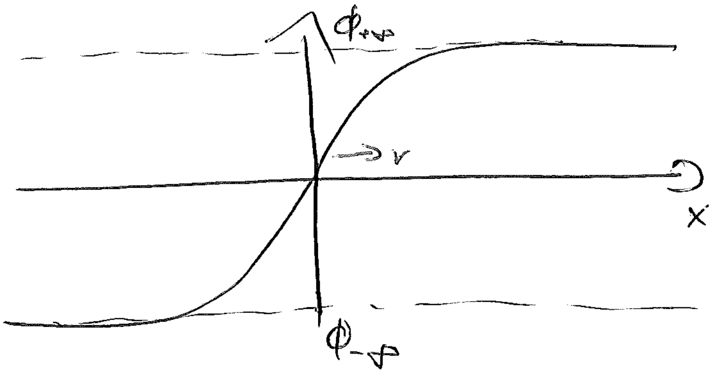
$$Q \simeq \int_{-\infty}^{+\infty} dx\, \partial_x \phi_0 = 2 \cdot \sqrt{6\mu^2/\lambda} \qquad (1.50)$$

(b) sin-Gordon Kink:

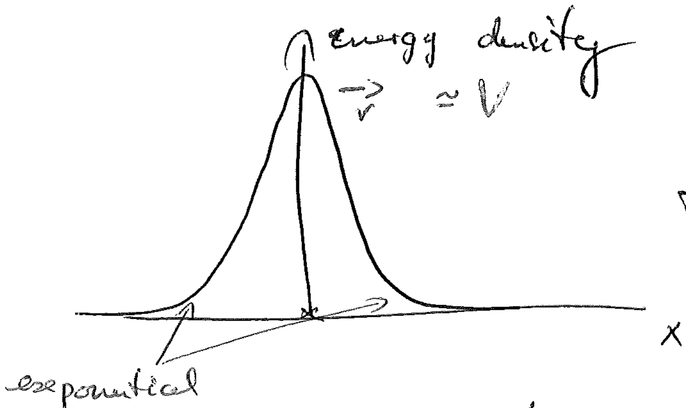
$$\phi_0(x) = 4/\lambda \, \arctan\left[e^{\mu(x-x_0)}\right]$$

$$Q \simeq \int_{-\infty}^{+\infty} dx\, \partial_x \phi_0 = 2\pi/\lambda \qquad (1.51)$$

2nd lecture



kink



energy density

fig 1

Energy of kink:

$$H_{static}[\phi_{kink}] = \frac{1}{2}\int dx\, \underbrace{\left(\partial_x \phi_{kink}\right)^2}_{=\,2V(\phi_{kink})} + \int dx\, V(\phi_{kink})$$

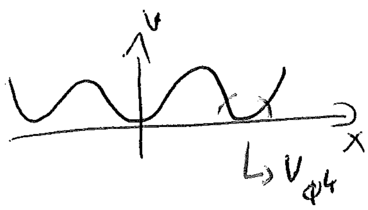
virial theorem $\quad = \int dx\, V(\phi_{kink})$

$$\underset{sEoM}{=} \int dx\, \frac{\partial \phi_{kink}}{\partial x}\sqrt{2V(\phi_{kink})} = \int_{\phi_{-\infty}}^{\phi_{+\infty}} d\phi\, \sqrt{2V(\phi)} \qquad (1.52)$$

asymptotics: ϕ^4-kink: $\phi_0(x \to \pm\infty) = \phi_{\pm\infty}\left(1 - e^{-2\mu/\sqrt{2}\,(|x|-x_0)}\right)$

$\Rightarrow$ fig 1

sine-Gordon: similarly:



ϕ^4: $\quad H[\phi_{kink}] = 4\cdot\sqrt{2}\,\mu^3/\lambda$ $\rightarrow$ see p. 18a

sine-Gordon: $\quad H[\phi_{kink}] = \dfrac{8\mu}{\lambda^2}$

'Non-static' solutions:

Lorentz Boost: $x \to \gamma(x - vt) = z$

$$\gamma = \frac{1}{\sqrt{1-v^2}} \qquad (1.53)$$

Kink with velocity v

solitary wave or soliton

Energy $E \to \gamma E \qquad (E = p^0)$

$$H[\phi] = \int dx \left\{ \frac{1}{2}\left[(\partial_{x_0}\phi)^2 + (\partial_x \phi)^2 \right] + V(\phi) \right\}$$

$$= \frac{1}{2}\underbrace{\left(\gamma^2 v^2 + \gamma^2 + 1\right)}_{2\gamma^2} \int \frac{dz}{\gamma} \left(\frac{\partial \phi}{\partial z}\right)^2$$

$$= \gamma \cdot \int dz \left(\frac{\partial \phi}{\partial z}\right)^2$$

sEoM $$= \gamma \cdot \int dz \, \frac{\partial \phi}{\partial z} \sqrt{2V(\phi(z))} = \gamma \int_{\phi_{-\infty}}^{\phi_{+\infty}} d\phi_0 \sqrt{2V(\phi_0)}$$

$$(1.54)$$

Proof of minimality of kink-solution

$$H[\phi] \geq H[\phi_{kink}]$$

Bogomol'nyi bound (BPS-trick):

$$H_{static} = \int dx \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi) \right]$$

$$= \frac{1}{2}\int dx \left[\frac{\partial\phi}{\partial x} \mp \sqrt{2V(\phi)} \right]^2 \pm \int dx \, \frac{\partial\phi}{\partial x}\sqrt{2V(\phi)}$$

$$= \underbrace{\frac{1}{2}\int \left[P\left(\phi, \frac{\partial\phi}{\partial x}\right) \right]^2}_{\geq 0} \pm \underbrace{\int_{\phi_{-\infty}}^{\phi_{+\infty}} d\phi \, \sqrt{2V(\phi)}}_{\text{top. term}} \tag{1.55}$$

$$P\left(\phi, \frac{\partial\phi}{\partial x}\right) := \frac{\partial\phi}{\partial x} \mp \sqrt{2V(\phi)}$$

vanishes for kink (anti-kink)
trivially for class. vacuum

$\Rightarrow$

$$\boxed{H_{static} \geq |\text{ top. term }|} \tag{1.56}$$

The bound is saturated if $P(\phi, \frac{\partial\phi}{\partial x}) = 0$

$$\boxed{P\left(\phi, \frac{\partial\phi}{\partial x}\right) = 0 \quad \text{for solitons}} \tag{1.57}$$

comp. topo. term:

$$\int_{\phi_{-\infty}}^{\phi_{+\infty}} d\phi \sqrt{2V(\phi)} = \int_{\phi_{-\infty}}^{\phi_{+\infty}} d\phi \, \frac{\partial W(\phi)}{\partial \phi} \tag{1.58}$$

with $\boxed{\frac{\partial W(\phi)}{\partial \phi} = \sqrt{2V(\phi)}}$

in ϕ^4-theory: $W(\phi) = \frac{\lambda}{12}\left(\frac{6\mu^2}{\lambda} - \phi^2\right)\phi$

for $|\phi| \leq \sqrt{\frac{6\mu^2}{\lambda}}$

$$\Rightarrow \pm \int_{\phi_{-\infty}}^{\phi_{+\infty}} d\phi \sqrt{2V(\phi)} = W \Big|_{\phi_-}^{\phi_+} = \sqrt{2} \cdot 4 \mu^3/\lambda \tag{1.59}$$

General case:

vacua

$$\ldots \phi_n < \phi_{n+1} < \ldots < \phi_{n+m} \ldots$$

connected by symmetry transformation

$V[\phi_n+\varphi] = V[\phi_{n+m}+\varphi]$

take finite energy config ϕ

with $\phi_{-\infty} = \phi_0$

$\phi_{+\infty} = \phi_n$

$$\Rightarrow H_{static}[\phi] = \frac{1}{2}\int_{\mathbb{R}} dx \left[\frac{\partial\phi}{\partial x} - \sqrt{2V(\phi)}\right]^2 + \int_{\phi_0}^{\phi_n} d\phi \sqrt{2V(\phi)}$$

$$\text{symmetry} \Rightarrow \quad = \frac{1}{2}\int_{\mathbb{R}} dx \left[\frac{\partial\phi}{\partial x} - \sqrt{2V(\phi)}\right]^2 + n\int_{\phi_0}^{\phi_1} d\phi \sqrt{2V(\phi)} \tag{1.60}$$

$$\Rightarrow \boxed{H_{static}[\phi] \geqslant n\int_{\phi_0}^{\phi_1} d\phi \sqrt{2V(\phi)}} = n\, H_{static}[\phi_{kink}] \tag{1.61}$$

Saturation: $\boxed{\dfrac{\partial\phi}{\partial x} = \sqrt{2V(\phi)}}$

Reminder: in sectors with $|Q_{top}| \geq 2$ no saturation

p 16 for explicit solution

But: $\inf\{H_{\text{stat}}[\phi]\} = n\, H_{\text{static}}[\phi_{\text{kink}}]$ (1.62)

Example: sine-Gordon model

kink: $\phi_{1;x_0}(x) = 4/\lambda \arctan[e^{\nu(x-x_0)}]$

Sector with $Q_{top} = n$:

$$\phi_{n;x_1\ldots x_n}(x) = \phi_{1;x_1}(x) + \ldots + \phi_{1;x_n}$$

$$Q_{top} = \frac{\lambda}{2\pi}\Big(\underbrace{\phi_{n;x_1\ldots x_n}(\infty)}_{n\cdot 2\pi/\lambda} - \underbrace{\phi_{n;x_1\ldots x_n}(-\infty)}_{0}\Big) = n \quad (1.63)$$

static energy: (i) $\left(\vec{\partial}\phi_{n;x_1\ldots x_n}\right)^2 = \sum_{i=1}^{n}\left(\vec{\partial}\phi_{i;x_i}\right)^2$

$$+ \sum_{i\neq j}\underbrace{\vec{\partial}\phi_{i;x_i}\vec{\partial}\phi_{j;j}}_{|x_i-x_j|\to\infty \quad \sim e^{-\nu|x_i-x_j|}} \quad (1.64a)$$

(ii) $V\left(\phi_{n;x_1\ldots x_n}(x)\right) = V[\phi_{1;x_{\hat{n}}}(x) + \varepsilon]$

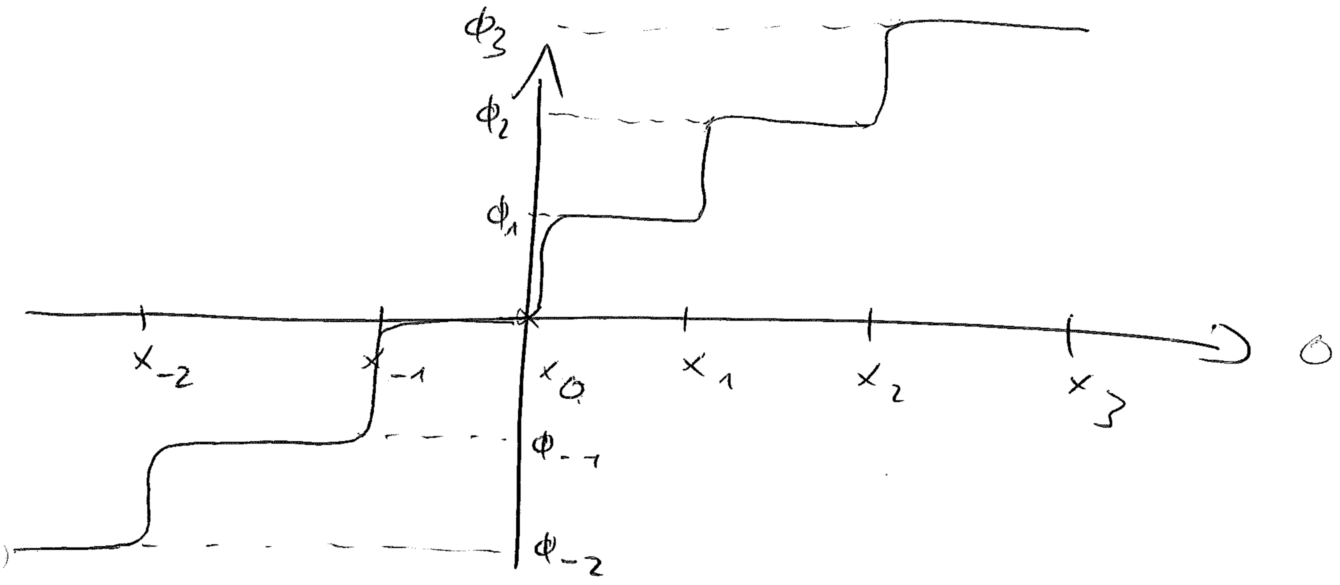
$$\varepsilon = \sum_{k=\hat{n}}\phi_{1,x_i}(x) \sim \sum_{i\neq\hat{n}} e^{-\nu|x-x_i|} \quad (1.64b)$$

with $\boxed{x_{\hat{n}-1} < x < x_{\hat{n}}}$

$$\Rightarrow \boxed{\lim_{|x_i - x_j| \to \infty} H_{static}[\phi_{n; x_1 \dots x_n}] = n\, H_{static}[\phi_1]} \quad (1.65)$$

$\Rightarrow$ dilute gas of kinks 'nearly' (exponentially) saturates the bound

Dilute gas approximation



Particle aspects of Solitons

Consider energy-momentum tensor $T^{\mu\nu}$

$$T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi + \eta^{\mu\nu}\left[\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi + V(\phi)\right] \tag{1.66}$$

with $(\eta^{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

4-momentum P^μ:

$$P^\mu = \int_{\mathbb{R}} dx\, T^{\mu 0} = (E, \vec{p}) \tag{1.67}$$

static soliton (at rest): $E = M$, M rest mass

localised: $\left(\frac{1}{2}(\vec{\partial}\phi)^2 + V(\phi)\right)$

$$M = \int_{\mathbb{R}} T^{00}(x)\, dx = H_{static}$$

$$\vec{p} = \int_{\mathbb{R}} T^{i0}\, dx = \int_{\mathbb{R}} \vec{\partial}\phi\, \underbrace{\partial^0\phi}_{0} = 0 \tag{1.68}$$

soliton with velocity v:

$$E = \gamma M, \quad \vec{p} = M\gamma v \tag{1.69}$$

See p. 17a $\sim$ relativistic point particle kink $\simeq$ particle

anti-kink $\simeq$ anti-particle

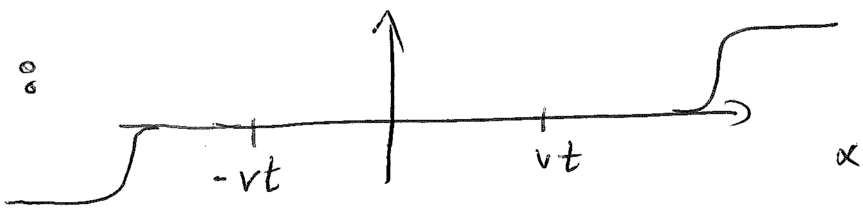
Construct kink–anti-kink (e^+e^-) solution ϕ_{KAK}
or kink-kink (e^-e^-) solution ϕ_{KK}

Example: sine-Gordon

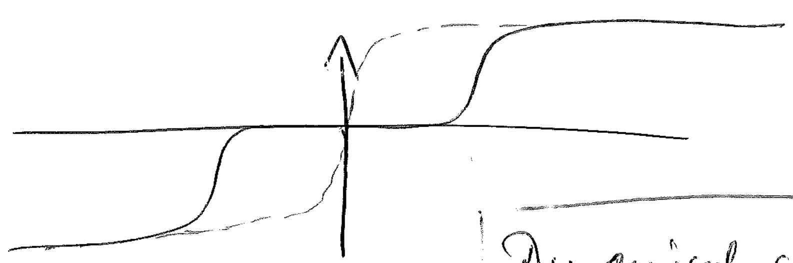
(a) scattering (e^-e^-) $\gamma = \frac{1}{\sqrt{1-v^2}}$, $Q_{top} = 2$

$$\phi_{KK} = 4/\lambda \arctan\left[v \frac{\sinh \mu\gamma x}{\cosh \mu\gamma v t} \right] \quad (1.76)$$

$t \to \pm\infty$:



finite t

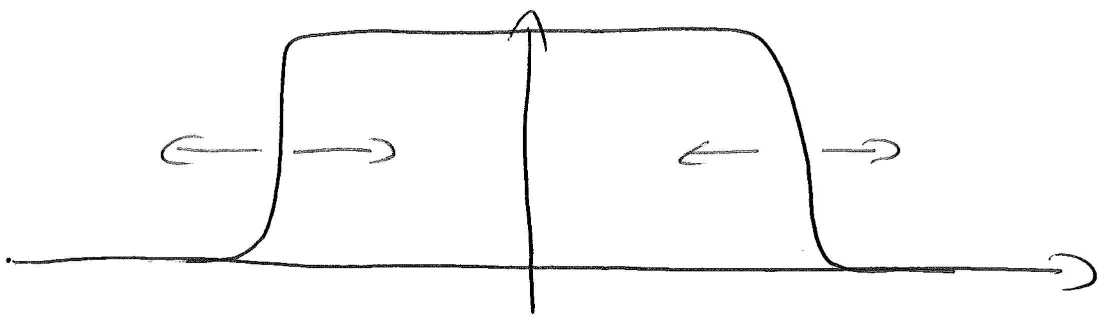


$\boxed{\text{Dynamical sol. of EoM}}$

(b) bound state ϕ_{KAK} 'breather', $Q_{top} = 0$

$$\phi_{KAK} = 4/\lambda \arctan 1/v \frac{\sin \mu \frac{vt}{\sqrt{1+v^2}}}{\cosh \mu \frac{x}{\sqrt{1+v^2}}} \qquad (1.71)$$

$$H[\phi_{KAK}] = \frac{2M}{\sqrt{1+v^2}} < 2M \qquad \text{for } v>0 \qquad (1.72)$$



$\boxed{\text{dynamical sol. of EoM}}$

Dynamical stability of Solitons

(1) topological stability: Q_{top} discrete

$$\dot{Q}_{top} = 0$$

'a soliton cannot decay'

$$H(\phi_{soliton}) = \min H(\phi) \quad \text{with } Q_{top}(\phi) = Q(\phi_{soliton})$$

(2.) $\phi_{soliton}$ has minimal energy

$$\Rightarrow \quad \phi = \phi_{soliton} + \lambda \varphi \tag{1.73}$$

static energy:

$$H[\phi_\lambda] = H[\phi_{soliton}] + \overbrace{\frac{1}{2}\lambda^2 \int_{\mathbb{R}} dx\, \varphi(x)\left[-\frac{\partial^2}{\partial x^2} + V''(\phi_{soliton})\right] \cdot \varphi(x)}^{\delta H}$$

$$+ \mathcal{O}(\lambda^3) \tag{1.74}$$

stability: $\boxed{\delta H \geq 0}$

$$\Leftrightarrow \quad -\frac{\partial^2}{\partial x^2} + V''(\phi_{soliton}) \quad \text{positive operator}$$

Eigenvalue problem:

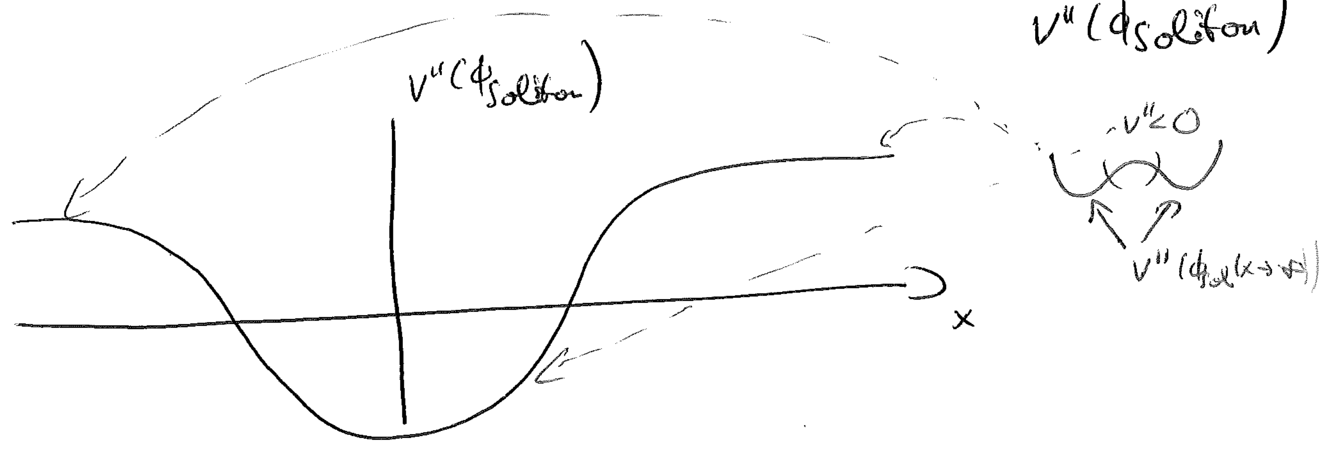
$$\left[-\frac{\partial^2}{\partial x^2} + V''(\phi_{\text{soliton}})\right]\varphi_n(x) = \omega_n^2\, \varphi_n(x) \qquad (1.75)$$

with $(\varphi_n, \varphi_m) = \int_{\mathbb{R}} dx\, \varphi_n(x)\varphi_m(x) = 1$

$$\Rightarrow \boxed{\delta H = \frac{1}{2}\sum_n (\lambda\, \omega_n)^2} \qquad (1.76)$$

If $\omega_n^2 \geq 0 \Rightarrow \delta H \geq 0$

(1.75) 'Schrödinger equation' with potential $V''(\phi_{\text{soliton}})$



For $\omega_n^2 \leq 0$ the knot theorem applies:
('bound states')

ground state: 0 knots

1st excited state: 1 knot

⋮

$\frac{1}{N_0}\frac{\partial \phi_{sol}}{\partial x} = \varphi_0$ eigenstate, with $\omega_0 = 0$ and 0 nodes:

$$\frac{\partial^2 \phi_{soliton}}{\partial x^2} = V'(\phi_{soliton})$$

$$\leadsto \quad \frac{\partial^3 \phi_{soliton}}{\partial x^3} = V''(\phi_{soliton})\frac{\partial \phi_{soliton}}{\partial x} \tag{1.77}$$

$$\Rightarrow \boxed{\left(-\frac{\partial^2}{\partial x^2} + V''(\phi_{soliton})\right)\frac{\partial \phi_{soliton}}{\partial x} = 0} \tag{1.78}$$

$$\omega_0^2 = 0$$

(1) $\phi_{soliton}$ monotonous $\Rightarrow$ $\varphi_0 \geq 0$

(2) $\varphi_0(x \to \pm\infty) \sim e^{-\nu|x|}$: φ_0 normalisable

$\Rightarrow$ φ_0 ground state : $\omega_n^2 > 0$ $n > 0$

dynamical stability

Remarks : zero mode φ_0 is related to translation invariance of $\phi_{soliton}$ generated by $\partial/\partial x$.

3rd lecture

Moduli : $\frac{\partial \phi_{1;x_1}}{\partial x} = \frac{\partial \phi_{1;x_1}}{\partial x_1}$ x_1 moduli parameter

- construction of solitons
- zero modes due to symmetries

normalisable zero modes ⇔ moduli

- dynamical solutions $\boxed{E_{dyn} \geq E_{stat}}$

What about higher dims?

Derrick's Theorem: (no solitons in $d>3$ in scalar theories)

static energy:

$$H_{static}[\phi^a] = \underbrace{\int d^{d-1}x \left[\tfrac{1}{2}(\vec{\partial}\phi^a)^2\right.}_{H_{kin}[\phi^a]} + \left.\underbrace{V(\phi^a)}_{\int d^{d-1}x}\right]$$

$$= H_{kin}[\phi^a] + H_{pot}[\phi^a] \qquad (1.79)$$

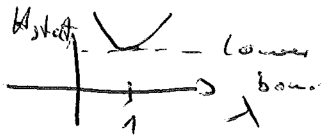
$\phi^a(x)$ static solution. Consider

$$\phi^a_\lambda(x) = \phi^a(\lambda x)$$

$$\Rightarrow H_{static}[\phi^a_\lambda] = \lambda^{1-d}\int d^{d-1}\lambda x\, \frac{\lambda^2}{2}\left(\frac{\partial\phi^a(\lambda x)}{\partial\lambda x_i}\right)^2 + \lambda^{1-d}\int d^{d-1}\lambda x \cdot V(\phi^a(\lambda x))$$

$$= \lambda^{3-d} H_{kin}[\phi^a] + \lambda^{1-d} H_{pot}[\phi^a] \qquad (1.80)$$

ϕ^a static solution: $\boxed{\left.\frac{\partial H}{\partial\lambda}\right|_{\lambda=1} = 0}$



$$\Rightarrow \left.\frac{\partial H}{\partial\lambda}\right|_{\lambda=1} = (3-d)\, H_{kin}[\phi^a] + (1-d)\, H_{pot}[\phi^a] = 0$$

$$\rightsquigarrow \underbrace{(3-d)\, H_{kin}[\phi^a]}_{\geq 0} = \underbrace{(d-1)\, H_{pot}[\phi^a]}_{\geq 0} \qquad (1.81)$$

$d>3$: LHS ≤ 0 , RHS ≥ 0 $\lightning$

$\Rightarrow$ any non-trivial solution $\phi^a \neq 0$

<u>instable</u>: no solitons

$d=3$: $$\boxed{H_{pot}[\phi^a] = 0} \qquad (1.82)$$

non-trivial sol. have to satisfy (1.82)

'exceptional models', not for $a=1$, real scalar

$d=2$: $$\boxed{H_{kin}[\phi^a] = H_{pot}[\phi^a]} \qquad (1.83)$$

non-trivial sol. have to satisfy

the virial theorem

$$\boxed{\frac{1}{2}\int \left(\frac{\partial \phi}{\partial x}\right)^2 = \int dx\, V(\phi)} \qquad (1.84)$$

indeed:

$$\left(\frac{\partial \phi_{kink}}{\partial x}\right)^2 = V(\phi_{kink}) \qquad (1.85)$$