1.2 Fermions in a solitonic background

$$\mathcal{L}_{total}[\phi,\psi] = \mathcal{L}_{bosonic}[\phi] + \mathcal{L}_{fermion}[\psi,\phi]$$

$$\mathcal{L}_{fermion}[\psi,\phi] = -\overline{\psi}(x)\left(i\gamma_\mu \partial^\mu + g\,\phi(x)\right)\psi(x) \qquad (1.86)$$

Hamiltonian: $\left(\pi = \frac{\partial \mathcal{L}}{\partial \partial_0 \psi}\right)$

$$H[\psi,\phi] = \gamma^0\left(\gamma^1 \frac{\partial}{\partial x} + g\,\phi\right)$$

with $\{H, \gamma^1\} = 0$ (1.87)

$$\Rightarrow \quad H\,\psi_n = E_n \psi_n \quad \Rightarrow H\gamma^1\psi_n = -E_n\gamma^1\psi_n$$

If $E_0 = 0$: H, γ^1 can be diagonalised simultaneously

$$S_\pm : \quad \gamma^1 S_\pm = \pm S_\pm$$

$$\Rightarrow \boxed{\psi_{0\pm}(x) = \frac{1}{N_\pm} e^{\mp g\int_0^x dy\,\phi(y)}\, S_\pm} \qquad (1.88)$$

$$\Rightarrow \quad H\varphi_{0\pm}(x) = \frac{1}{N_\pm}\gamma^0\left(\mp g\phi(x)\gamma^1 s_\mp + g\phi s_\pm\right) \tag{1.88}$$

$$= 0$$

Remarks:

(1) $\varphi_{0\pm}$ is exponentially normalised for $Q_{top.} = \pm 1$: one normalised zero mode

(2) Index theorem (Bott-Seeley) determines (differences) of # of norm. ferm. zero modes