1.3 Dilute gas expansion

static approximation in quantum mechanical path integral

$\phi(x) \sim x(t)$

$$G_\pm(x) = \frac{1}{N} \int_{\phi(0)=\phi_\pm}^{\phi(\,)=\phi_+} d\phi(x)\; e^{-S[\phi]/\hbar}$$

Euclidean

$$\simeq \frac{1}{N}\int d\phi(x) \sum_{\text{Solutions } S} e^{-\frac{1}{\hbar}\left(S[\phi_s] + \frac{1}{2}\int_{x,x'} \underbrace{\frac{\delta^2 S}{\delta\phi^2}\Big|_{\phi_s}}_{S^{(2)}[\phi_s]} (\phi-\phi_s)^2\right)}$$

saddle-point approx.

$$\frac{\delta S}{\delta \phi}\Big|_{\phi_s} = 0$$

$$\boxed{S^{(2)} \cdot \phi_{0S} = 0}$$

$$\simeq \sum_S \int \frac{d\phi_{0S}}{N} \underbrace{\int d\phi'(x)\; e^{-\frac{1}{\hbar}S[\phi_s]} \cdot e^{-\frac{1}{2\hbar}\int_{x,x'} \phi' S^{(2)} \phi'}}_{\text{Gaußian}}$$

$$\downarrow$$

$$\simeq \frac{1}{N'} \sum_S \frac{\prod \int dx_i \;\; J_S}{\sqrt{\det' S^{(2)}[\phi_s]}}\, e^{-\frac{1}{\hbar}S[\phi_s]}$$

with $S^{(2)} \simeq \left[-\partial_x^2 + V''(\phi_s)\right]$

Dilute gas: Φ_s : n solitons, n / $n-1$ anti-solitons

$$\sqrt{\det' S^{(2)}[\Phi_s]} = \underbrace{e^{-\omega x/2}}_{\substack{\text{trivial vac} \\ \text{'}\hbar\omega/2\text{'}}} \cdot K^{2n} \;/ (K^{2n-1})$$

$$e^{-\frac{1}{\hbar}S[\Phi_s]} \simeq e^{-\frac{2n}{\hbar}S[\Phi_{Q=1}]}$$
$$/ e^{-\frac{2n-1}{\hbar}S[\Phi_{Q=1}]}$$

$$\int \prod_i dx_i \; J_s = J^{2n} / J^{2n-1} \cdot x^n / n!$$

$$\Rightarrow G_\pm(x) = \frac{1}{N'} e^{-\omega x/2} \sum_{n_\pm} \frac{x^{n_\pm}}{n_\pm!} J^{n_\pm} K^{n_\pm} e^{-n_\pm/\hbar\, S[\Phi_{Q=1}]}$$

$$n_+ = 2n$$
$$n_- = 2n-1$$

$$= \boxed{\frac{1}{N'} e^{-\frac{\omega x}{2}} \left[e^{xJKe^{-S[\Phi_1]/\hbar}} \pm e^{-xJKe^{-S[\Phi_1]/\hbar}} \right]}$$

$$G_\pm(x) = \psi_0^*(\Phi_\pm)\,(\Phi_+)\, e^{-E_0 x/\hbar} + \psi_1^*(\Phi_\pm)\psi_1(\Phi_\pm) \cdot e^{-E_1 x/\hbar}$$

$$\Rightarrow \boxed{E_{0,1} = \hbar\omega/2 \mp \hbar J K e^{-S[\Phi_1]/\hbar}}$$