2 Gauge theories & homotopy

In 2-dim scalar theories: classification of

$$\partial M_{space} \to M_{vacuum} \qquad (2.1)$$

In higher dimensions: no solitons in scalar theories (Derrick's theorem)
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but singular 'solutions' (with infinite energy)

Consider compl. scalar theory in 2+1 dim:

$$\mathcal{L} = -\partial_\nu \phi^* \partial^\nu \phi - \underbrace{\frac{1}{2}\left(\phi^*\phi - v^2\right)^2}_{V(\phi)} \qquad (2.2)$$

classification of maps

$$\begin{array}{ccc} \partial M_{space} & \longrightarrow & \{\phi \mid \phi^*\phi = v^2\} \\ \wr\wr & & \wr\wr \\ S^1 & \longrightarrow & S^1 \end{array} \qquad (2.3)$$

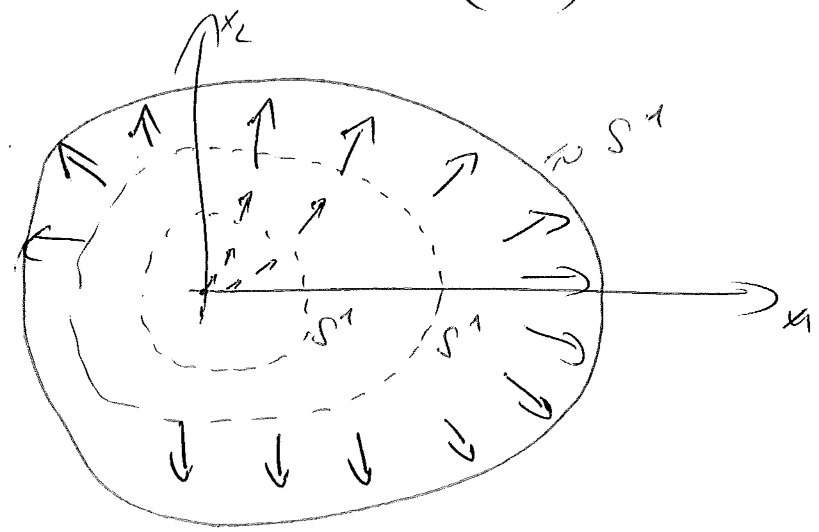
$\Rightarrow$ Winding number: $n \in \mathbb{Z}$ 'wrapping' S^1 around S^1

'solution' $\phi = \phi_1 + i\phi_2$ with

$$\lim_{|\vec{x}|\to\infty} \phi_i = v \frac{x_i}{\underbrace{\sqrt{x_1^2+x_2^2}}_{|\vec{x}|}} + \mathcal{O}(1/|\vec{x}|^2) \qquad (2.4)$$

$$\Rightarrow \quad \lim_{|\vec{x}|\to\infty} V'(\phi) = 0 + \mathcal{O}(1/|\vec{x}|^2)$$

$$\lim_{|\vec{x}|\to\infty} \vec{\partial}^2 \phi_i = -\frac{x_i}{(x_1^2+x_2^2)^{3/2}} + \mathcal{O}(1/|\vec{x}|^3) \qquad (2.5)$$



$\Rightarrow$ ϕ must be zero somewhere (for ϕ being smooth)

Energy:

$$H_{static} = \int d^2x \left[\vec{\partial}\phi \vec{\partial}\phi^* + V(\phi) \right]$$

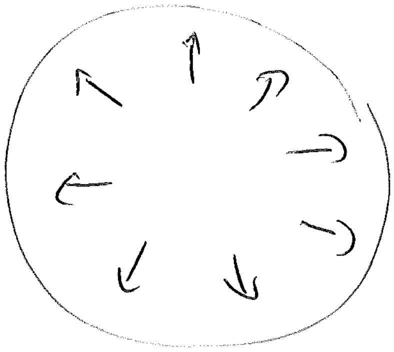
$$\simeq \lim_{R\to\infty} \int_{|\vec{x}|\le R} d^2x \left[\frac{v^2}{x_1^2+x_2^2} + \mathcal{O}\left(\tfrac{1}{|\vec{x}|^3}\right) \right] \qquad (2.6)$$

$$\partial_i \phi_j = \left(\delta_{ij} - \frac{x_i x_j}{|\vec{x}|^2}\right) \frac{v}{|\vec{x}|}$$

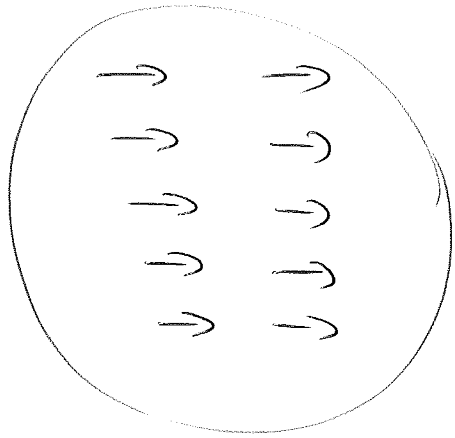
$\sim \ln R \to \infty$

[H_{stat} div. $\Leftrightarrow$ Derrick's theorem]

Topological charge:



vs



Write

$$\phi(x) = u(x)\, v$$

Then

$$Q_{top} = \frac{i}{2\pi}\int_{\partial M} (u\, d u^\dagger) \qquad \leftarrow (\phi^{-1} d\phi)$$

$$= \frac{i}{2\pi}\int_{\partial M \cong S^1} d\vec{s}\; u \cdot \vec{\partial} u^\dagger$$

$$= \frac{i}{2\pi}\int_0^{2\pi} d\Theta\; u\, \partial_\Theta u^\dagger \qquad u\big|_{\partial M} = e^{+i\varphi(\Theta)}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\Theta\, \frac{\partial\varphi}{\partial\Theta} = \frac{\varphi(2\pi) - \varphi(0)}{2\pi} = n$$