2.1 Abelian Higgs model (2+1 dim)

Gauge $U(1)$-symmetry: local $U(1)$

$$\partial_\nu \phi \to D_\nu \phi = \left(\partial_\nu + i e A_\nu\right) \phi \qquad (2.7)$$

$$\mathcal{L} = -\frac{1}{4} F_{\nu\nu} F^{\nu\nu} - (D_\nu \phi)^* D^\nu \phi - V(\phi) \qquad (2.8)$$

with

$$F_{\nu\nu} = \partial_\nu A_\nu - \partial_\nu A_\nu = \frac{1}{ie}\left[D_\nu, D_\nu\right] \qquad (2.9)$$

Gauge invariance $\quad U = e^{i\omega} \in U(1)$

$$\begin{aligned} \phi^U(x) &= U(x)\,\phi(x) \\ A^U_\nu(x) &= A_\nu(x) + \frac{1}{ie} U(x)\partial_\nu U^+(x) \\ &= \frac{1}{ie} U(x)\, D_\nu\, U^+(x) \end{aligned} \qquad (2.10)$$

$$\Rightarrow \qquad S[A^U, \phi^U] = S[A, \phi] \qquad (2.11)$$

with $S[A,\phi] = \int d^3x\, \mathcal{L}(A,\phi)$

EoM: $S = \int d^3x\, \mathcal{L}$

$$\frac{\delta S}{\delta A_\nu} = 0 : \quad \boxed{\partial_\mu F^{\mu\nu} = J^\nu} \qquad (2.12)$$

with $$J^\nu = ie(\phi\, \partial^\nu \phi^* - \phi^* \partial^\nu \phi) + 2e^2 \phi^* \phi A_\nu$$

$$\left[= ie\left(\phi (D^\nu \phi)^* - \phi^* D^\nu \phi\right)\right]$$

$$\frac{\delta S}{\delta \phi^*} = 0 : \quad \boxed{D_\mu D^\mu \phi = \frac{\partial V}{\partial \phi^*}} \qquad (2.13)$$

Recall topological scalar excitation: (static)

$$\phi_i \xrightarrow{|\vec{x}| \to \infty} v \frac{x_i}{|\vec{x}|} \; ; \quad \vec{\partial} \phi^* \vec{\partial} \phi = \frac{v^2}{|\vec{x}|^2} \qquad (2.14)$$

Gauge field $\vec{A}$ with

$$|\vec{x}|^2 (\vec{D}\phi)^* \vec{D}\phi \xrightarrow{|\vec{x}| \to \infty} 0$$

$$A_i \xrightarrow{|\vec{x}| \to \infty} \frac{1}{ie} \partial_i \ln \phi_{as} = \underbrace{\frac{1}{ie} \mathcal{U}\, \partial_i \mathcal{U}^+}_{\text{pure gauge}} + \mathcal{O}(1/|\vec{x}|)$$

with

$$\phi_{as} = \mathcal{U}^+(\vec{x}) \cdot v + \mathcal{O}(1/|\vec{x}|) \qquad (2.15)$$

$$\Rightarrow |\vec{x}^2| \cdot F_{12}^2(A) \xrightarrow{|\vec{x}| \to \infty} 0$$

$$|\vec{x}|^2 (\vec{D}\phi)^* \vec{D}\phi \xrightarrow{|\vec{x}| \to \infty} 0 \qquad (2.16)$$

$$\Rightarrow \boxed{H_{stat} = \int d^2x \left[(\vec{D}\phi)^* \vec{D}\phi + \tfrac{1}{2} F_{12}^2 + V(\phi) \right] < \infty}$$

(2.17)

With $\vec{A}_{as} = \frac{1}{e} \vec{\partial} \Theta$ with $x_1 = r\cos\Theta$, $x_2 = r\sin\Theta$

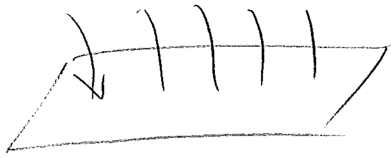
$$= \frac{1}{e}\frac{1}{r}\hat{e}_\Theta \quad \text{with } \hat{e}_\Theta = \begin{pmatrix} -\sin\Theta \\ \cos\Theta \end{pmatrix} \qquad (2.18)$$

$\Rightarrow$ topological charge:

magnetic field strength $\left[B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk} = (\vec{\partial} \times \vec{A})_i \right]$ (2.19)

$$B = F_{12} \leftarrow B_3$$

magnetic flux:



$$\Phi = \int_{\mathcal{M}} d^2x\, F_{12} \underset{\text{stokes}}{=} \int_{\partial\mathcal{M}} d\vec{S} \cdot \vec{A} = \frac{1}{e}\int_0^{2\pi} d\Theta\, r \frac{1}{r} \partial_\Theta \Theta$$

$d\vec{S} = r\,\hat{e}_\Theta\, d\Theta$

$$= \frac{1}{e} \cdot 2\pi \qquad (2.20)$$

4th lecture

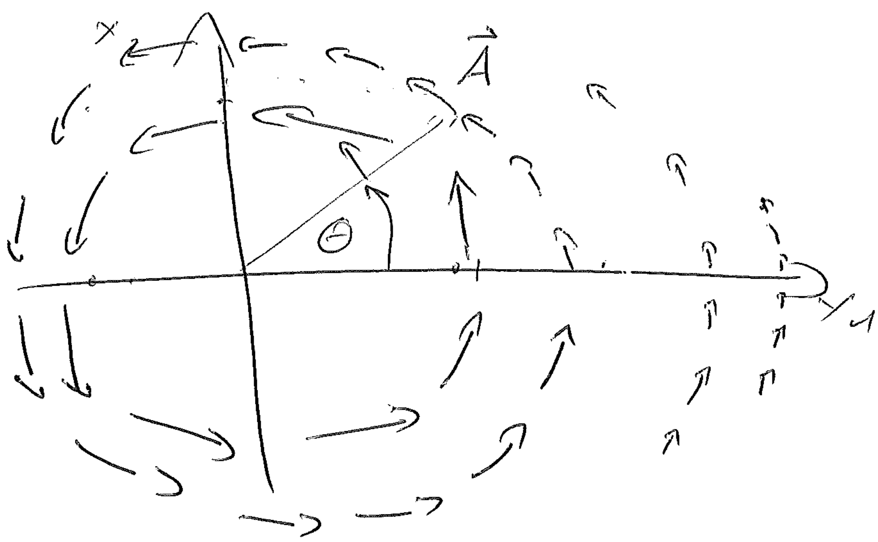
in general $\boxed{\Phi = \frac{1}{e} 2\pi n}$ ← Winding number $S^1 \to S^1$ (2.21)

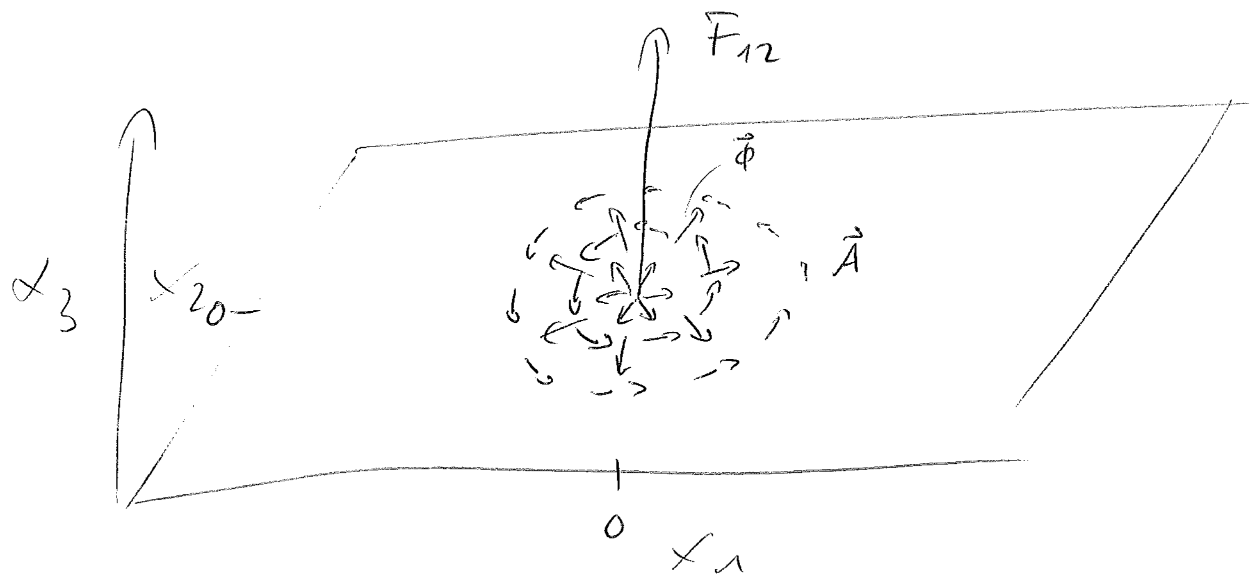
Let $\vec{A}(\vec{x}) = \vec{A}_{as}(\vec{x}) \quad \Rightarrow \quad F_{12} = 0$

$$= \frac{1}{e}\frac{1}{r}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \frac{1}{e}\frac{1}{r^2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

$$\Rightarrow \partial_1 A_2 - \partial_2 A_1 = \frac{1}{e}\left(\partial_1 \frac{x_1}{x_1^2+x_2^2} + \partial_2 \frac{x_2}{x_1^2+x_2^2}\right) = 0$$

but :





Bogomol'nyi bound:

write energy as sum of squares + top. term:

$$(\vec{D}\phi)^* \vec{D}\phi = |(D_1 \pm i D_2)\phi|^2 \mp e \underbrace{F_{12}}_{B_3} \phi^*\phi$$

+ total deriv.

(2.22)

$$\frac{1}{2}F_{12}^2 + \frac{\lambda}{2}(\phi^*\phi - v^2)^2$$

$$= \frac{1}{2}\left(F_{12} \mp \sqrt{\lambda}(\phi^*\phi - v^2)\right)^2 \pm \sqrt{\lambda}\, F_{12}(\phi^*\phi - v^2)$$

(2.23)

$$\Rightarrow \quad H_{stat} = \int d^2x \Big\{ |(D_1 \pm i D_2)\phi|^2 + \frac{1}{2}\left(F_{12} \mp \sqrt{\lambda}(\phi^*\phi - v^2)\right)^2$$

$$\pm(\sqrt{\lambda} - e) F_{12}\, \phi^*\phi \; \underbrace{\boxed{\mp \sqrt{\lambda}\, v^2 F_{12}}}_{\text{topol. term}} \Big\}$$

critical coupling $\boxed{\lambda = e^2}$

(2.24)

$$\boxed{H_{stat} \geq \left| \sqrt{\lambda}\, v^2 \int d^2x\, F_{12} \right|} \qquad (2.25)$$

Saturation: $(\lambda = e^2)$

$$\boxed{\begin{aligned} F_{12} &= \pm e(\phi^*\phi - v^2) \\ (\mathcal{D}_1 \pm i\mathcal{D}_2)\phi &= 0 \end{aligned}} \qquad (2.26)$$

$\lambda = e^2$: $s F \circ \hat{M}$

Remarks: (1) relevant for superconductivity

$\lambda > e^2$: Type I (vortex free)

$\lambda < e^2$: Type II (vortex excitations)

(2) Characterised by

$$S^1 \longrightarrow \underset{\underset{U(1)}{\uparrow}}{S^1}$$

$\Rightarrow$ classification of continuous maps $\quad \llcorner$ gauge group

$\Rightarrow$ homotopy

in general

$$\partial M_{\text{space}} \longrightarrow (S)U(N), \; SO(10) \ldots$$

$d = 1+1$	S
$d = 2+1$	$S^2, T^2, \ldots$
$d = 3+1$	$S^3, S \times \mathbb{R}^2, S \times T^2, T^3$
$\vdots$	Tori, orbifolds, ...