2.2 Homotopy

Motivation: Classification of maps f

$$f: \partial\mathcal{M}_{space} \longrightarrow \mathcal{M}_{vacuum}$$

$\Rightarrow$ Homotopy (equivalent modulo continuous deformations)

2.2.1 Fundamental group π_1

Definition of homotopy:

Let X, Y be smooth manifolds and f is a smooth map $f: X \to Y$

Homotopy F:

$$F: X \times I \longrightarrow Y \qquad , I = [0,1]$$

with
$$F(\underset{\in X}{x}, \underset{\in I}{0}) = f(x) \tag{2.27}$$

$f_t(x) = F(x,t)$ are homotopic to each other, in particular to $f_0 = f$

in general
f, g are homotopic, $f \sim g$, if they can be continuously deformed into each other

e.g. $f, g: \mathbb{R}^n \to \mathbb{R}^n$! $\mathbb{R}^n$ open !

$$f(x) = x \quad , \quad g(x) = x_0 \tag{2.28}$$

Homotopy F:

$$F(x,t) = (1-t)x + t x_0$$

Remark: Spaces X, where the identity map $\mathbb{1}_X$ and the constant map are homotopic, are called contractible.

Homotopically equivalent:

Definition: Spaces X and Y are homotopically equivalent if continuous maps f, g exist

$$f: X \to Y \quad , \quad g: Y \to X$$

$$g \circ f \sim \mathbb{1}_X \quad , \quad f \circ g \sim \mathbb{1}_Y \tag{2.29}$$

Example:

$$S^n = \{ \vec{x} \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1 \} \sim \mathbb{R}^{n+1} \setminus \{0\} \qquad (2.30)$$

Definition of connectedness:

Definition of loop: closed path through x_0 in X.

$$\alpha : I \to X \quad \text{with} \quad \alpha(0) = \alpha(1) = x_0$$

Product of loops: (def. of algebraic structure)

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} \qquad (2.31a)$$

inverse loop

$$\alpha^{-1}(t) = \alpha(1-t) \qquad (2.31b)$$

constant loop

$$c_{x_0}(t) = x_0 \qquad (2.31c)$$

Question: Is $c_{x_0}(t)$ unit element? No.

$\alpha_{x_0} * \alpha_{x_0}^{-1} \neq c_{x_0}$, but $\alpha_{x_0} * \alpha_{x_0}^{-1} \sim c_{x_0} \to$ Equiv. classes

Definition: two loops $\alpha_{x_0}, \beta_{x_0}$ are homotopic, $\alpha_{x_0} \sim \beta_{x_0}$, if H exists with

$$H : I \times I \to X \qquad (2.32)$$

with

$$F(s,0) = \alpha_{x_0}(s) \;, \quad 0 \le s \le 1$$

$$F(s,1) = \beta_{x_0}(s) \;, \quad 0 \le s \le 1$$

and $\quad F(0,t) = F(1,t) = x_0 \quad 0 \le t \le 1 \quad$ x_0 fixed

$$(2.33)$$

Definition of fundamental group π_1

$\pi_1(X, x_0)$ is the set of equivalence classes (homotopy classes) $[\alpha]$ of loops through x_0.

with

$$[\alpha_{x_0}] * [\beta_{x_0}] := [\alpha_{x_0} * \beta_{x_0}] \qquad (2.34)$$

Remarks: (i) • reflexivity of $\sim$
- symmetry
- transitivity

$\Rightarrow$ group structure: (1) $([\alpha]*[\beta])*[\gamma] = [\alpha]*([\beta]*[\gamma])$

(2) $[\alpha_x]*[c_x] = [\alpha_x] = [c_x]*[\alpha_x]$

(3) $[\alpha]*\underbrace{[\alpha^{-1}]}_{[\alpha]^{-1}} = [c_x]$

(2.35)

(ii) X arc wise connected

$$\Rightarrow \boxed{\pi_1(X, x_0) \overset{\text{isomorphic}}{\cong} \pi_1(X, x_1)} \qquad \forall x_0 x_1 \in X$$

(2.36)

$\Rightarrow \quad = \pi_1[X]$

(iii) The fundamental groups of homotopically equivalent spaces are isomorphic: $X \sim Y$

$$\Rightarrow \boxed{\pi_1[X] \cong \pi_1[Y]} \qquad \text{(arcwise connected)}$$

(2.37)

$$\boxed{\pi_1[X, x_0] \cong \pi_1[Y, f(x_0)]} \qquad \text{in general}$$

(2.38)

$f: X \to Y$ $\qquad f \circ g \sim 1\!\!1_Y, \; g \circ f \sim 1\!\!1_X$

$g: Y \to X$

Examples & extensions

1) Def.: A topological space X is simply connected, if any loop in X can be continuously shrunk to a point

$$\Rightarrow \pi_1(X) = 0 \qquad (2.39)$$

e.g. $\mathbb{R}^n$, counterexamples, S^1, $T^2 = S^1 \times S^1, \dots, T^n$

2.) Abelian Higgs model

$$\begin{array}{ccc} \partial M_{space} & \rightarrow & M_{vacuum} \\ \wr\!\wr & & \wr\!\wr \\ S^1 & \longrightarrow & S^1 \quad (U(1)) \end{array}$$

$$[\quad S^1 \quad \rightarrow \quad U(1) \quad \text{gauge group} \quad]$$

$$(2.40)$$

$$\Rightarrow \quad \pi_1(S^1) = \mathbb{Z}$$

Map χ: $S^1 \rightarrow S^1$

$$e^{i\theta} \rightarrow e^{i\chi(\theta)}$$

or $\theta \rightarrow \chi(\theta)$ in the Lie-algebra

$$(2.41)$$

Summary:

Def of fundamental group $\pi_1(X)$

via equivalence classes of maps α

$$\alpha : [0,1] \longrightarrow X$$

$$\alpha(0) = \alpha(1)$$

$$\Rightarrow \alpha : S^1 \rightarrow X$$

Def. of algebr. structure

$\alpha * \beta$ see page 43, eq. (2.31)

α^{-1}, e = const. loop

Ex.:

$\pi_1(S^1) = \mathbb{Z}$ p. 46

Map:
$$F(\Theta, t) = (1-t)\,\mathcal{X}(\Theta) + t\,\Theta\,\frac{\mathcal{X}(2\pi)}{2\pi}$$

with
$$F(0,t) = \mathcal{X}(0) = 0 \qquad (2.42)$$
$$F(2\pi, t) = \mathcal{X}(2\pi)$$

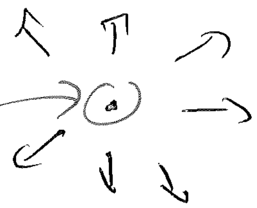
is homotopy

$$\Rightarrow \boxed{\mathcal{X}(\Theta) \sim m\,\Theta}$$

m ← Integers $\Rightarrow \pi_1(S^1) \cong \mathbb{Z}$

$\hookleftarrow$ def. (2.43)

Vortex:



$\Rightarrow$ defined on $\mathbb{R}^2 \setminus \{0\}$
$\wr$
S^1

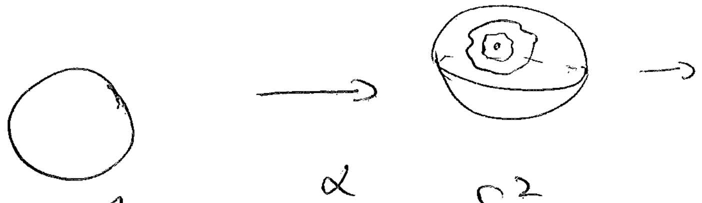
singular point $\Theta(0)$ not well-defined

is stable, as it cannot decay into
$([\alpha] \simeq 1)$
the vacuum $([\alpha] \sim 0)$.

higher dimensions: $\mathbb{R}^n \setminus \{0\} \sim S^{n-1}$ point-defect

e.g. $\mathbb{R}^3 \setminus \{0\} \sim S^2$

Loop on S^2



$$S^1 \xrightarrow{\alpha} S^2$$

$$[\alpha] \cong 0 \qquad \forall \alpha$$

$\Rightarrow \quad \pi_1(S^2) = 0$

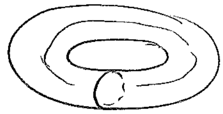
general $\quad \pi_1(S^n) = 0 \qquad n \geq 2$ $\qquad$ (2.43)

n-spheres with $n > 1$ are simply connected

X, Y arcwise connected

$\Rightarrow \pi_1(X \times Y) \simeq \pi_1(X) \oplus \pi_1(Y)$ $\qquad$ (2.44)

e.g. torus $T^2 = S^1 \times S^1$



$\rightarrow$ p. 51

$\pi_1(T^2) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \oplus \pi_1(S^1) = \mathbb{Z} \oplus \mathbb{Z}$

4.) $SU(2)$ weak gauge group, QCD - 'model' theory

$$(2.45)$$

$u \in SU(2)$: $\quad u = a_0 \sigma_0 + i \vec{\sigma}\vec{a} \quad$ with $a_0^2 + \vec{a}^2 = 1$

$$u u^\dagger = a_0^2 + \vec{a}^2 = 1$$

$\sigma_0 = \mathbb{1}_2$

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Lie-algebra

$\boxed{[\sigma_i, \sigma_j] = i\varepsilon_{ijk}\sigma_k}$

$$\Rightarrow SU(2) \simeq S^3$$

$$\boxed{\pi_1(SU(2)) = \{e\}}$$

5.) $SO(3)$ rotations in $\mathbb{R}^3$: $\vec{x} \to R\vec{x} = \vec{x}$

$R \in SO(3) \quad R R^T = \mathbb{1}$ characterised by

Lie-algebra $\boxed{[L_i, L_j] = i\varepsilon_{ijk}L_k}$ Euler-angles

Alternatively: $\hat{x} = \vec{x}\cdot\vec{\sigma}$, $\hat{R} = a_0(R)\sigma_0 + i\vec{\sigma}\vec{a}(R)$ $\in SU(2)$

$$\hat{x}^R = \hat{R}^\dagger \hat{x} \hat{R} \qquad (2.46)$$

constitutes a rotation of $\vec{x}$:

$$x_i^R = \mathrm{Tr}\, \sigma_i \hat{x}^R \qquad (2.47)$$

$\Rightarrow$ Exercise: Translation in Euler-angles

$SL(2,\mathbb{C})$ vs $SO(1,3)$

However: $\hat{R}$, $\pm\hat{R}$ induce the same rotation

$$\hat{x}^{\pm R} = \hat{x}^{R} \left(= (\pm)^2 \hat{R}^\dagger \hat{x} \hat{R} \right) \qquad (2.48)$$

$\Rightarrow$ $SO(3) \simeq SU(2)/\mathbb{Z}_2$ $\leftarrow$ center of $SU(2)$

Center of G: $\{u \in G \mid u^{-1} g u = g \ \forall g \in G\}$

$$\Rightarrow \pi_1(SO(3)) = \pi_1(SU(2)/\mathbb{Z}_2) = \mathbb{Z}_2$$

$$\left[\qquad = \pi_0(\mathbb{Z}_2) = \mathbb{Z}_2 \quad \right]$$

connectedness

$$(2.49)$$

$\Rightarrow$ higher (lower) homotopy groups
topological groups

$SU(2)$ is the universal covering group of $SO(3)$

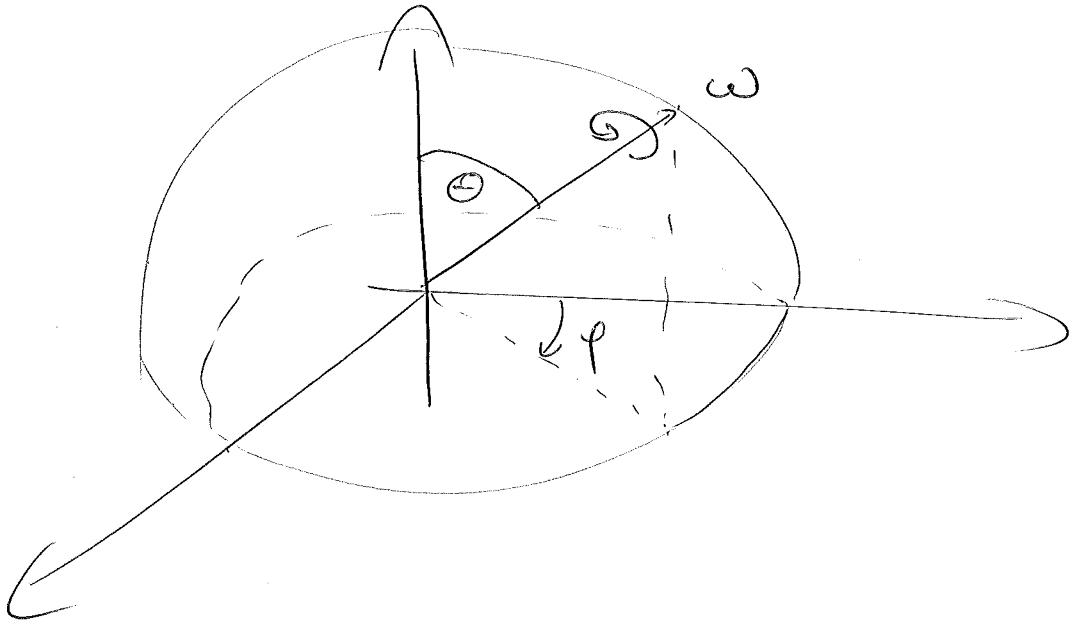
Def.: $(\tilde{X}, f)$ is covering space of X if

(1) $f: \tilde{X} \to X$ surjective

(2) for each $x \in X$ there is an open set $x \ni U \subset X$ with $p^{-1}(U)$ is disjoint union of open sets in $\tilde{X}$, each homeomorphic to U

Universal covering space: $\tilde{X}$ simply connected

S^3



$\varphi \in [0, 2\pi]$

$\Theta \in [0, \pi/2]$

$\omega \in [-\pi/2, \pi/2]$

Remarks

Univ. Covering groups of $SO(n)$: Spin groups

$Spin(SO(3)) \cong SU(2)$

$Spin(SO(4)) \simeq SU(2) \times SU(2)$

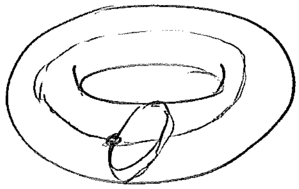
unitarily equivalent

In Minkowski space: univ. covering group of Lorentzgr.

$SL(2, \mathbb{C})$

6.) Fundamental group of tori:

$$T^n = \underbrace{S \times S \times \dots \times S}_{n\text{-times}}$$

$T^2 = S \times S$ 

2 fundamental cycles

$$\pi_1(T^2) \simeq \pi_1(S) \oplus \pi_1(S) = \mathbb{Z} \oplus \mathbb{Z}$$

$$\pi_1(T^n) \simeq \pi_1(S) \oplus \dots \oplus \pi_1(S) = \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$$

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in general:

$$\pi_1(X \times Y) = \pi_1(X) \oplus \pi_1(Y)$$

see p. 48
eq. (2.44)