2.2.2 Higher homotopy groups π_n

Motivation: characterisation of topo. in non-Abelian gauge theories e.g. SU(2) on $\mathbb{R}^4$ with 'boundary' $\partial \mathbb{R}^4 \simeq S^3$

$\Rightarrow$ maps:

$$\begin{array}{ccc} \partial M_{\text{space-time}} & \longrightarrow & SU(2) \\ \wr\wr & & \wr\wr \\ S^3 & \longrightarrow & S^3 \end{array} \qquad (2.50)$$

more generally $S^n \longrightarrow$ gauge group

$\left[n=1 \quad I=[0,1] \xrightarrow{\alpha} \text{gauge group} \right.$ with $\left. \alpha(0)=\alpha(1) \right]$

We define

$$I^n = \{ (s_1, \ldots, s_n) \mid 0 \leq s_i \leq 1 \quad \forall i \}$$

$$\partial I^n = \{ (s_1, \ldots, s_n) \in I^n \mid s_i = 0 \wedge s_i = 1 \text{ for at least one } i \}$$

$$\simeq S^{n-1} \qquad (2.51)$$

$$\alpha_{x_0} : I^n \longrightarrow X \quad \text{with} \quad \alpha_{x_0}(\vec{s}) = x_0 \quad \text{for } \vec{s} \in \partial I^n \tag{2.52}$$

$[\underbrace{I^n/\partial I^n}_{\cong S^n} \rightarrow X]$

homotopy F:

$$F : I^n \times I \longrightarrow X \tag{2.53}$$

with

$$F(s_1, \ldots, s_n, 0) = \alpha(s_1, \ldots, s_n)$$

$$F(s_1, \ldots, s_n, 0) = \beta(s_1, \ldots, s_n)$$

$$F(s_1, \ldots, s_n, t) = x_0 \quad \text{for } \vec{s} \in \partial I^n \tag{2.54}$$

If F exists $\Rightarrow$ $\boxed{\alpha \sim \beta}$

Product of n-loops:

$$\alpha * \beta(s_1, \ldots, s_n) = \begin{cases} \alpha(2s_1, s_2, \ldots, s_n), & 0 \leq s_1 \leq 1/2 \\ \alpha(2s_1 - 1, s_2, \ldots, s_n), & 1/2 \leq s_1 \leq 1 \end{cases} \tag{2.55a}$$

Inverse

$$\alpha^{-1}(s_1, \ldots, s_n) = \alpha(1 - s_1, s_2, \ldots, s_n) \tag{2.55b}$$

unit element $\quad e(s_1, \ldots, s_n) = x_0 \quad$ (2.55c) nth homotopy group: (arcwise connected X)

$\pi_n(X)$ is the set of equivalence classes (homotopy classes) $[\alpha]$ of n-loops with algebraic structure $*$:

$$[\alpha] * [\beta] := [\alpha * \beta] \qquad (2.56)$$

Remark: $\pi_n(X)$ is Abelian for $n > 1$

Examples & extensions

1) Maps $S^m \to S^n$: $\pi_m(S^n)$

(i) $\pi_n(S^n) \simeq \mathbb{Z}$

e.g. $n=2$: Θ, φ polar coordinates
θ', φ'

$$\begin{aligned} \Theta' &= \Theta \\ \varphi' &= m\varphi \end{aligned} \qquad (2.57)$$

Exercise: reduce all maps to the standard m-map

$\pi_n(X)$: classifies maps from $S^n \to X$

or $I^n/\partial I^n \to X$

Important example: $\pi_n(S^n) = \mathbb{Z}$

$$\boxed{\pi_3(SU(2)) \simeq \pi_3(S^3) = \mathbb{Z}}$$

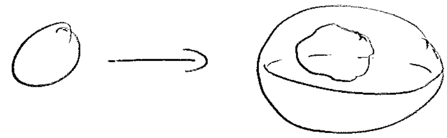
or $\pi_2(S^2)$:

Θ, φ polar coord.

$\Theta' = \Theta$

$\varphi' = m\varphi$

(ii) $\pi_m(S^n) = 0 \quad \text{for } m < n \qquad (2.58)$

e.g. $\pi_1(S^n)$: 

6th lecture

2.) Hopf-invariant: $\pi_3(S^2) \simeq \mathbb{Z}$

$$S^3 \xrightarrow{\pi} S^2$$

(Dirac) - magnetic monopoles

see p. 55a

$$S^3 \simeq \{ \vec{x} \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \} \qquad (2.59)$$

complex coordinates

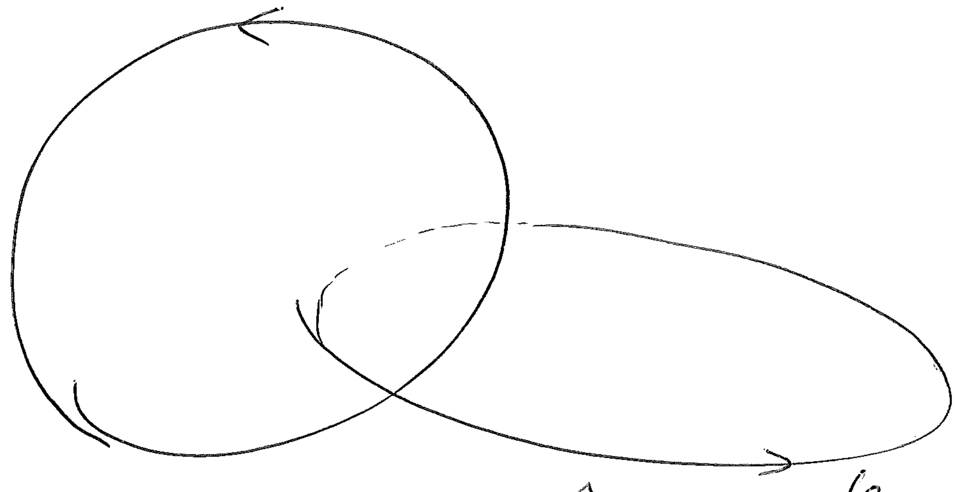
$$z_1 = x_1 + i x_2 \,, \quad z_2 = x_3 + i x_4$$
$$\bar{z}_1 = x_1 - i x_2 \qquad \bar{z}_2 = x_3 - i x_4 \qquad (2.60)$$
$$\Rightarrow \quad z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1$$

$$S^2 \simeq \{ \vec{y} \in \mathbb{R}^3 \mid y_1^2 + y_2^2 + y_3^2 = 1 \} \qquad (2.61)$$

Hopf map: $\pi : S^3 \to S^2$

$$\pi : \quad y_1 = 2(x_1 x_3 + x_2 x_4) = 2 \,\mathrm{Re}\, z_1 \bar{z}_2$$
$$y_2 = 2(x_2 x_3 - x_1 x_4) = 2 \,\mathrm{Im}\, z_1 \bar{z}_2$$
$$y_3 = z_1 \bar{z}_1 - z_2 \bar{z}_2 \qquad (2.62)$$

Work W of magnetic monopole:



$$W = g \cdot \int_{\mathcal{C}_1} \vec{B}(s_1)\, d\vec{s}_1$$

$\mu_0/4\pi \to 1$

$$\vec{B}(s_1) = I \int_{\mathcal{C}_2} d\vec{s}_2 \times \frac{\vec{s}_1 - \vec{s}_2}{\|\vec{s}_1 - \vec{s}_2\|^3} \longleftarrow d\vec{B} = I\,\frac{d\vec{s} \times \vec{x}}{\|\vec{x}\|^3}$$

$$\Rightarrow W = g \cdot I \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} d\vec{s}_2 \times \frac{\vec{s}_1 - \vec{s}_2}{\|\vec{s}_1 - \vec{s}_2\|^3} \cdot d\vec{s}_1$$

$$= g \cdot I \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \frac{d\vec{s}_1 \times d\vec{s}_2}{\|\vec{s}_1 - \vec{s}_2\|^2} \, \frac{\vec{s}_1 - \vec{s}_2}{\|\vec{s}_1 - \vec{s}_2\|}$$

$$[\vec{y}(\vec{x})]^2 = [\vec{x}^2]^2 = 1 \qquad (2.63)$$

$$\Longleftrightarrow \quad \vec{y}(\vec{x}) \in S^2 \quad \text{(embedded in } \mathbb{R}^3\text{)}$$

Question: inverse image (Urbild) of a point $\vec{y} \in S^2$?

We write

$$y = (2 z_1 \bar{z}_2 \,,\; z_1\bar{z}_1 - z_2\bar{z}_2) \qquad (2.64)$$

$$\Rightarrow \quad \lambda z_i \quad \text{with} \quad \lambda \in \{\lambda \,|\, \lambda\bar{\lambda} = 1 \,|\, \lambda \in \mathbb{C}\} \simeq S^1 \simeq U(1)$$

with $\boxed{y(z_1, z_2) = y(\lambda z_1, \lambda z_2)}$ (2.65)

Also $(z_1', z_2') = (\lambda z_1, \lambda z_2)$ are <u>all</u> (z_1', z_2') with $y(z_1,z_2) = y(z_1', z_2')$

$$\Rightarrow \quad \pi^{-1}(\vec{y}) \simeq U(1) \qquad (2.66)$$

It follows $\boxed{\pi_3(S^2) = \mathbb{Z}}$ $[\pi] \simeq 1$ (2.67)

Remark: $\Rightarrow$ S^3 is principle $U(1)$ - bundle over S^2: $(S^3, \pi, S^2, U(1))$

total space, project., base space, fibre, structure group

$$U(1) \hookrightarrow S^3 \xrightarrow{\pi} S^2 \qquad (2.68)$$

$\pi^{-1}(S^2) = U(1)$

Exercise: local trivialisation

locally $S^3 \simeq S^2 \times U(1)$

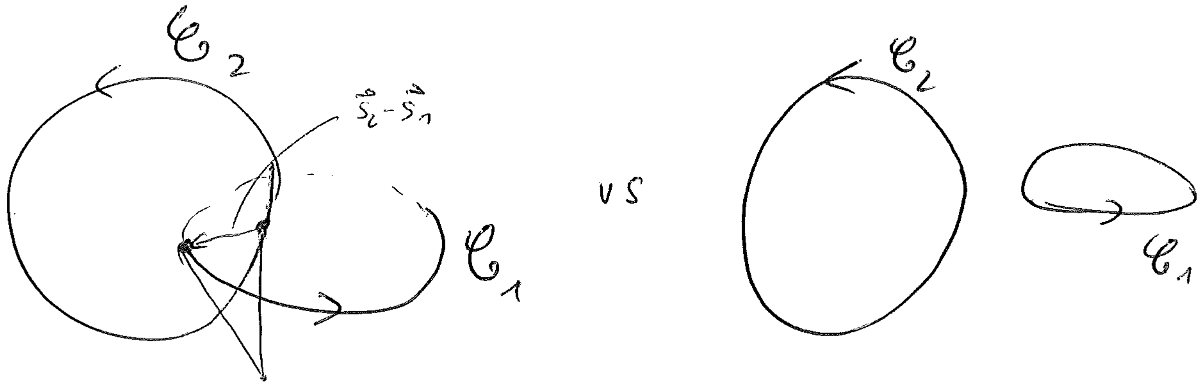
Geometrical interpretation

$$\pi^{-1}(y_i) = \mathcal{C}_i \subset S^3 \qquad (2.69)$$

$\mathcal{C}_i$ closed curves on S^3 (or better) $\underset{\mathcal{C}_i}{S^1} \times S^2$

described by $(\lambda z_1, \lambda z_2)$ (in $\mathbb{R}^4$)

position of two curves $\mathcal{C}_1, \mathcal{C}_2$



Linking number: $lk(\mathcal{C}_1, \mathcal{C}_2)$ (curves in $\mathbb{R}^3$)

'how many times does $\mathcal{C}_1$ wind around $\mathcal{C}_2$'

(or vice versa)

$$lk(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{4\pi} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{d\vec{s}_1 \times d\vec{s}_2}{\|\vec{s}_2 - \vec{s}_1\|^2} \frac{\vec{s}_2 - \vec{s}_1}{\|\vec{s}_2 - \vec{s}_1\|}$$

(2.70)

$lk(\mathcal{C}_1, \mathcal{C}_2)$ cannot be changed by continuous deformations!

$$\Rightarrow \quad lk(\mathcal{C}_1, \mathcal{C}_2) \text{ labels } m \in \mathbb{Z} \simeq \pi_3(S^2)$$

Moreover: $lk(\mathcal{C}_1, \mathcal{C}_2)$ classifies maps

Gauß map

$$T^2 \longrightarrow S^2$$

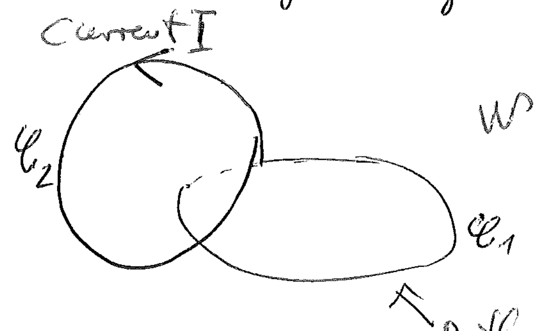
$$(t_1, t_2) \xrightarrow{f} \frac{\vec{S}_1(t_1) - \vec{S}_2(t_2)}{\| \vec{S}_1(t_1) - \vec{S}_2(t_2) \|} = y \in S^2 \tag{2.71}$$

lk measures degree of the map f

"$\deg f$ = weighted # of inverse images of $y \in S^2$"

Remark: work W of magnetic monopole

see p. 55a



$$W = g \int_{\mathcal{C}_1} \vec{B}(s_1)\, d\vec{s}_1 = \frac{4\pi g}{(c)} I \; lk(\mathcal{C}_1, \mathcal{C}_2)$$

path of magn. monopole with charge g.

3.) degree of a map: X, Y closed oriented manifolds of same dim m

$$f : X \to Y$$

Jacobian J: $y_0 \in Y$ with inverse Image

$$f^{-1}(y_0) = \{x_1, \dots, x_m\}$$

$$J_i = \det \frac{\partial y_0^\alpha}{\partial x_i^\beta}$$

where y_0^α, x_i^β are local coordinates of y_0, x_i (2.72)

If y_0 is regular: $J_i \neq 0 \quad \forall\, i = 1, \dots, m$

$\Rightarrow$ degree of f in y_0:

$$\boxed{\deg f = \sum_{x_i \in f^{-1}(y_0)} \operatorname{sgn}(J_i)} \qquad (2.73)$$

Properties:

(1) $\deg f$ indep. of reg. point y_0

(2) $\deg f$ invariant under homotopies

$\Rightarrow$ classification of homotopy classes

e.g.: $f, g \quad X \to S^n$ are homotopic

iff $\deg f = \deg g$ (2.74)

example: $S^1 \to S^1$

$$\deg f = \frac{1}{2\pi} \int_0^{2\pi} dx \, \frac{df}{dx} \qquad (2.75)$$

in general: $\left[(f^* \omega)(x) = \omega(f(x)) \; , x \in X \right]$

$X \xrightarrow{f} Y$

$$\int_X f^* \omega = \deg f \int_Y \omega \qquad (2.76)$$

$\uparrow$ pull back

$\mathbb{R}^n$:

$$\int_{f^{-1}(U_i)} \omega(y(x)) \det \frac{\partial y^\alpha}{\partial x_i{}^\beta} \, dx_1 \dots dx_n$$

$$= \operatorname{sgn} \det \frac{\partial y^\alpha}{\partial x_i{}^\beta} \int_{U_i} \omega(y) \, dy_1 \dots dy_n$$

(2.77)

Hopf invariant:

Ω_n volume form of S^n, normalised to

$$\int_{S^n} \Omega_n = 1$$

$$f: S^{2n-1} \to S^n \qquad \begin{matrix}(S^3 \to S^2)\\ \text{for } n=2\end{matrix}$$

Write

$$\underbrace{f^* \Omega_n}_{\text{closed}} = d\,\omega_{n-1}$$

$$\boxed{H(f) = \int_{S^{2n-1}} \omega_{n-1} \wedge d\,\omega_{n-1}}$$

Exercise: compute $H(f)$ for Hopf map π, defined in (2.62)

Summary:

Hopf-invariant / map:

$$S^3 \xrightarrow{\pi} S^2$$

$$z = (z_1, z_2) \in \mathbb{C}^2 \longrightarrow (y_1, y_2, y_3) = y$$

$$z_1\bar{z}_1 + z_2\bar{z}_2 = 1$$

$$y_1 = 2\,\mathrm{Re}\, z_1\bar{z}_2$$

$$y_2 = 2\,\mathrm{Im}\, z_1\bar{z}_2$$

$$y_3 = z_1\bar{z}_1 - z_2\bar{z}_2$$

$$\Rightarrow \pi^{-1}(y) = \{\lambda z \quad \text{with} \quad \lambda \in \mathbb{C} \wedge \lambda\bar{\lambda} = 1\}$$

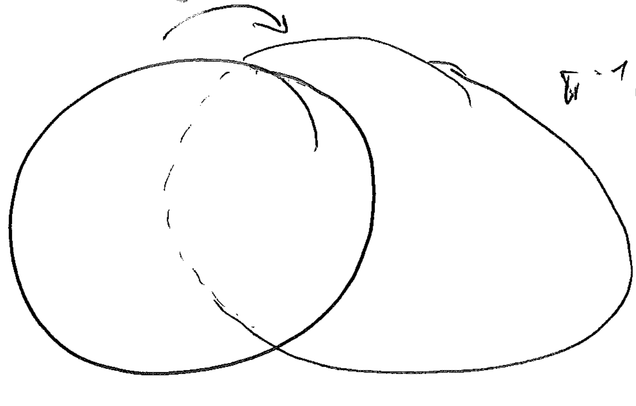
$$\simeq S^1 \simeq U(1)$$

locally: $S^3 \simeq S^2 \times U(1)$

(Follows also from $S^3 = I^4 / \partial I^4$)

$y \to y + \varepsilon$: S^1 lifted

$\pi^{-1}(y+\varepsilon)$



$\pi^{-1}(y)$

$$\Rightarrow lk\left(\pi^{-1}(y), \pi^{-1}(y+\varepsilon)\right)$$

π : standard map in homotopy class 1

$\Rightarrow$ classification

lk defined in (2.70) p. 57 for curves in $\mathbb{R}^3$

$$\sim \frac{\vec{S}_1(t_1) - S_2(t_2)}{\| \vec{S}_1(t_1) - \vec{S}_2(t_2) \|}$$