3 Non-Abelian gauge theories

3.1 Setting

non-Abelian gauge field (SU(N))

connection

$$A_\nu(x) = A_\nu^a\, t^a \quad \in su(N) \text{ Lie algebra}$$

generators

$a = 1, \ldots, N^2-1$

with Lie algebra

$$[t^a, t^b] = f^{abc}\, t^c$$

structure constants

for the generators t^a.

$$\mathrm{tr}_{\text{fund.}}\, t^a t^b = -\frac{1}{2}\delta^{ab}$$

Example: $SU(2)$: $\quad t^a = -i\sigma^a/2 \quad , f^{abc} = \varepsilon^{abc}$

see p. 49

Covariant derivative:

$$D_\nu = \partial_\nu + g A_\nu$$

transforms covariantly (as a tensor) under gauge transformations.

$\mathcal{U}(x) \in SU(N)$:

$$D_\nu^{\mathcal{U}} = \mathcal{U} D_\nu \mathcal{U}^+$$

$$A_\nu^{\mathcal{U}} = \frac{1}{g}(\mathcal{U} D_\nu \mathcal{U}^+) = \mathcal{U} A_\nu \mathcal{U}^+ + \frac{1}{g}\mathcal{U}\partial_\nu \mathcal{U}^+$$

Infinitesimal: $\mathcal{U} = e^{g\alpha} \simeq \mathbb{1} + g\alpha$

$$\Rightarrow D_\nu^{\mathcal{U}} = -g[D_\nu, \alpha] + \mathcal{O}(\alpha^2)$$

$$A_\nu^{\mathcal{U}} = A_\nu - [D_\nu, \alpha] + \mathcal{O}(\alpha^2)$$

Field strength (curvature):

$$F_{\nu\sigma} = \frac{1}{g}[D_\nu, D_\sigma]$$

7th lecture

$$F_{\nu\sigma}^{\mathcal{U}} = \frac{1}{g}\mathcal{U}[D_\nu \mathcal{U}^+, \mathcal{U} D_\nu]\mathcal{U}^+$$

$$= \mathcal{U} F_{\nu\sigma} \mathcal{U}^+$$

Action: (Euclidean)

$$S_{YM}[A] = \frac{1}{2}\int d^dx \ \mathrm{tr}(F^2)_{\nu\nu}$$

$$S_{YM}[A] = \frac{1}{2}\int d^d x \; \mathrm{tr}\; F^a_{\mu\nu} F^b_{\nu\mu} \; t^a t^b$$

$$= \frac{1}{4}\int d^d x \; F^a_{\mu\nu} F^a_{\mu\nu}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{bca} A^b A^c$

Evidently $S[A^u] = S[A]$

EoM: $\delta S = 0$

$$\frac{\delta S}{\delta A^a_\rho(y)} = \int d^d x \; \mathrm{tr}\; F_{\mu\nu} \frac{\delta F_{\nu\mu}}{\delta A_\rho}$$

$$= 2\int d^d x \; \mathrm{tr}\; F_{\mu\nu}(x)\left[D_\nu(x), t^a \delta(x-y)\right]$$

$$= -2\int d^d x \; \mathrm{tr}\left[D_\nu, F_{\mu\nu}\right] t^a \delta(x-y)$$

$$= 2[D_\nu, F_{\nu\mu}]^a$$

Matter: S_{Matter}

QCD: $S_{\text{quark}} = i\int d^d x \; \bar\psi \gamma_\mu D_\mu \psi$

with $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$

$$\frac{\delta S_{\text{quark}}}{\delta A^a_\rho(x)} = j^a_\rho(x) = i\bar\psi\gamma_\rho t^a \psi$$

Georgi - Glashow: $S_\phi = \frac{1}{2} \int d^dx \, (D_\nu \phi)^a (D_\nu \phi)^a$

$$(D_\nu \phi)^a = [D_\nu, \phi]^a$$

$$\frac{\delta S_\phi}{\delta A^a_\rho(x)} = j^a_\rho(x) = g f^{abc} \phi^b (D_\rho \phi)^c$$

$$\Rightarrow \quad \boxed{D_\mu F_{\mu\nu} = j_\nu}$$

inhomogeneous EoM

Bianchi-id: $\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} = \frac{1}{2g} \varepsilon_{\mu\nu\rho\sigma} [D_\rho, D_\sigma]$

$$\Rightarrow [D_\mu, \tilde{F}_{\mu\nu}] \simeq \left([D_\mu, [D_\rho, D_\sigma]] + \text{cyclic-perm} \right) \cdot \varepsilon_{\mu\nu\rho\sigma}$$

homogeneous EoM

$$= 0$$

$\uparrow$ integrability

It follows

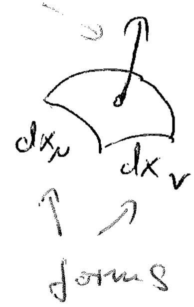
$$S_{top}[A] = \frac{1}{2} \int d^dx \, \mathrm{tr}\, F \tilde{F}$$

$$= \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \int d^dx \, \mathrm{tr}\, [D_\mu, D_\nu][D_\rho, D_\sigma]$$

$$= \int d^dx \, (\text{total deriv.}) = 0$$

Interlude: Forms on Manifolds M

consider 2-dim surface element $d\vec{F}$



wedge product: totally anti-symm tensor product of forms

$$dx_\mu \wedge dx_\nu = - dx_\nu \wedge dx_\mu$$

reflects direction of surface-element

Generally:

$$dx_{\mu_1} \wedge \dots \wedge dx_{\mu_i} \wedge dx_{\mu_{i+1}} \wedge \dots \wedge dx_{\mu_n}$$

$$= - dx_{\mu_1} \wedge \dots \wedge dx_{\mu_{i+1}} \wedge dx_{\mu_i} \dots \wedge dx_{\mu_n}$$

n- Form:

$$\omega = \frac{1}{n!} \omega_{\mu_1 \dots \mu_n} dx_{\mu_1} \wedge \dots \wedge dx_{\mu_n}$$

$\Rightarrow \omega_{\mu_1 \dots \mu_n}$ totally antisym.

Remarks: $\quad dx_{\mu_1} \wedge \dots \wedge dx_\nu \wedge \dots dx_\nu \wedge \dots dx_{\mu_n} = 0$

$n > \dim M \quad dx_{\mu_1} \wedge \dots \wedge dx_{\mu_n} = 0$

d : n-Forms $\rightarrow$ n+1-Forms

exterior deriv.

$$\omega \rightarrow d\omega = \frac{1}{n!} \frac{\partial \omega_{\mu_1 \cdots \mu_n}}{\partial x_{\mu_{n+1}}} dx_{\mu_{n+1}} \wedge dx_{\mu_1} \wedge \cdots \wedge dx_{\mu_n}$$

de Rham compl.

It follows $d^2 \omega = 0$

nomenclature: $d\omega = 0 \Rightarrow \omega$ is closed

$\omega = d\nu$ ω is (locally) exact

Exact forms are closed (when are closed forms exact? Poincaré lemma)

Stokes (from Poincaré's lemma)

$$\boxed{\int_{\mathcal{M}} d\omega = \int_{\partial \mathcal{M}} \omega}$$

∂ boundary op homology

e.g. $\int_a^b df = \int_{\partial[a,b]} f = f(b) - f(a)$ f: 0-Form

Application: F field strength is 2-form

$$F = \frac{1}{2} F_{\mu\nu} dx_\mu \wedge dx_\nu$$

$$= dA + g\, A \wedge A$$

1-form with $A = A_\mu dx_\mu$

$g=1$: or absorbed in A

$$F = \partial_\mu A_\nu \, dx_\mu \wedge dx_\nu + A_\mu A_\nu \, dx_\mu \wedge dx_\nu$$

$$= \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]\right) dx_\mu \wedge dx_\nu$$

4-dim (show)

$$F \wedge F = \frac{1}{4} \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \, d^4x \quad = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$$

$$\sim F_{\mu\nu} \tilde{F}_{\mu\nu} \, d^4x$$

Also:

$$\operatorname{tr} F \wedge F = \operatorname{tr}\left((dA + A^2)(dA + A^2)\right)$$

$$= \operatorname{tr}\left(dA\,dA + dA\,A^2 + A^2 dA + \underset{\overset{\|}{0}}{A^4}\right)$$

$$= \operatorname{tr}\left(d(A\,dA) + \frac{2}{3} d(A^3)\right)$$

$$= d \underbrace{\operatorname{tr}\left(A\,dA + \frac{2}{3} A^3\right)}_{\text{Chern-Simons form } K}$$

$$\Rightarrow \boxed{\int_M \operatorname{tr} F \wedge F = \int_M dK = \int_{\partial M} K}$$

$$\operatorname{tr} A^4 = 0$$

$$= \operatorname{tr} A_\mu A_\nu A_\rho A_\sigma \, dx_\mu \wedge \dots \wedge dx_\sigma$$

$$= \operatorname{tr} A_\sigma A_\mu A_\nu A_\rho \, dx_\mu \wedge \dots \wedge dx_\sigma$$

$$= - \operatorname{tr} A_\sigma A_\mu A_\nu A_\rho \, dx_\sigma \wedge dx_\mu \wedge dx_\nu \wedge dx_\rho$$

$$= - \operatorname{tr} A^4$$

Summary:

$$S_{YM}[A] = \frac{1}{2g^2}\int d^d x \, \mathrm{tr}\, F^2 = -\frac{1}{2g^2}\int \mathrm{tr}\, F \wedge * F$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}[\mathcal{D}_\rho, \mathcal{D}_\sigma]$$

$$\mathcal{D}_\mu = \partial_\mu + A_\mu$$

Forms:

$$F = \frac{1}{2} F_{\mu\nu}\, dx_\mu \wedge dx_\nu$$

$$\mathrm{tr}\, F \wedge F = \mathrm{tr}\,(dA + A^2)(dA + A^2)$$

$$= d\, \mathrm{tr}\left(A\, dA + \tfrac{2}{3} A^3\right)$$

and $$\mathrm{tr}\, \tilde{F}_{\mu\nu}^2 = \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\lambda\gamma}\, \mathrm{tr}\, F_{\rho\sigma} F_{\lambda\gamma}$$

$$= \mathrm{tr}\, F_{\rho\sigma}^2$$

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or $$\mathrm{tr}\, *F \wedge **F = \mathrm{tr}\, *F \wedge F.$$

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