3.2 Instantons in SU(2)

(anti-) self-dual configurations
homogeneous field equations (no current)

$$D_\mu F_{\mu\nu} = 0$$

$$D_\mu \tilde{F}_{\mu\nu} = 0 \qquad \leftarrow \text{Bianchi Id p. 65}$$

$$\Rightarrow D_\mu \left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu} \right) = 0$$

Remember Bogomol'nyi bound (p. 39)

$$\mathrm{tr}\left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu} \right)^2 = \mathrm{tr}\, F_{\mu\nu}^2 + \mathrm{tr}\, \tilde{F}_{\mu\nu}^2 \pm 2\,\mathrm{tr}\, F_{\mu\nu} \tilde{F}_{\mu\nu}$$

$$= 2\left(\mathrm{tr}\, F_{\mu\nu}^2 \pm \mathrm{tr}\, F_{\mu\nu} \tilde{F}_{\mu\nu} \right)$$

see p. 65 $\sim \mathrm{tr}\, F \wedge F$

↳ total deriv.

$$\text{with } \mathrm{tr}\, \tilde{F}_{\mu\nu}^2 = \frac{1}{4} \underbrace{\varepsilon_{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\lambda\gamma}}_{2(\delta_{\rho\lambda}\delta_{\sigma\gamma} - \delta_{\rho\gamma}\delta_{\sigma\lambda})} \mathrm{tr}\, F_{\rho\sigma} F_{\lambda\gamma}$$

$$= \frac{1}{2}\left(\mathrm{tr}\, F_{\rho\sigma} F_{\rho\sigma} - \mathrm{tr}\, F_{\rho\sigma} F_{\sigma\rho} \right)$$

$$= \mathrm{tr}\, F_{\rho\sigma}^2$$

$$\Rightarrow \quad \boxed{\operatorname{tr} F_{\mu\nu}^2 \geq \mp \operatorname{tr} F_{\mu\nu} \tilde{F}_{\mu\nu}}$$

and implies (p. 64)

$$\boxed{S_{YM}[A] \geq \pm \frac{1}{2} \int d^d x \operatorname{tr} F_{\mu\nu} \tilde{F}_{\mu\nu}} \qquad *$$

Equality for (anti-) self dual configurations

$$\boxed{F_{\mu\nu} = \mp \tilde{F}_{\mu\nu}}$$

Remarks: (1) self-dual configs are solutions to the EoM:

$$D_\mu F_{\mu\nu} = \pm D_\mu \tilde{F}_{\mu\nu} \overset{\text{Bianchi-Id}}{=} 0$$

(2) $\int d^d x \operatorname{tr} F_{\mu\nu} \tilde{F}_{\mu\nu}$ candidate for top-invariant, in which case $*$ serves as a bound for configs. with the given top. 'charge'.

9^{th} lecture

Computation on $\mathcal{M}^{d=4} = S^4$: $A \to g A$

$$-\frac{1}{2}\int_{S^4} d^4x \, \mathrm{tr}\, F_{\mu\nu} \tilde{F}_{\mu\nu}$$

$$= -\frac{1}{4}\int_{S^4} d^4x \, \varepsilon_{\mu\nu\rho\sigma} \, \mathrm{tr}\, F_{\mu\nu} F_{\rho\sigma}$$

cyclicity

$$\varepsilon_{\mu\nu\rho\sigma} \, \mathrm{tr}\, A_\mu A_\nu A_\rho A_\sigma = 0$$

$$= -\frac{1}{4}\int_{S^4} d^4x \, \varepsilon_{\mu\nu\rho\sigma}\left(4\,\mathrm{tr}\, \partial_\mu A_\nu \partial_\rho A_\sigma + 8\,\mathrm{tr}\, \partial_\mu A_\nu A_\rho A_\sigma\right)$$

$$= -\int_{S^4} d^4x \, \partial_\mu \underbrace{\varepsilon_{\mu\nu\rho\sigma} \, \mathrm{tr}\left(A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma\right)}_{-K_\mu}$$

If K is globally defined, then

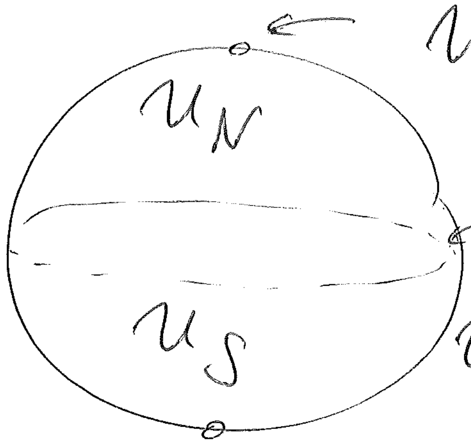
$$\boxed{-\frac{1}{2}\int_{S^4} d^4x \, \mathrm{tr}\, F_{\mu\nu} \tilde{F}_{\mu\nu} = 0}$$

vacuum sector

$$\boxed{S_A[A] \geq 0}$$

General configurations with smooth $\mathrm{tr}\, F^2$ (finite action)

S^4 :



$\mathcal{U}_N$: northern hemisphere

equator $S^3 \simeq \mathcal{U}_N \cap \mathcal{U}_S$

$\mathcal{U}_S$: southern hemisphere

e.g. S^4 one-point compactification of $\mathbb{R}^4$: N: infinity

S: 0 (stereographic proj.)

We have two patches

$\mathbb{R}^4 \to \mathcal{U}_N$ and gauge field $A_{N\mu}$

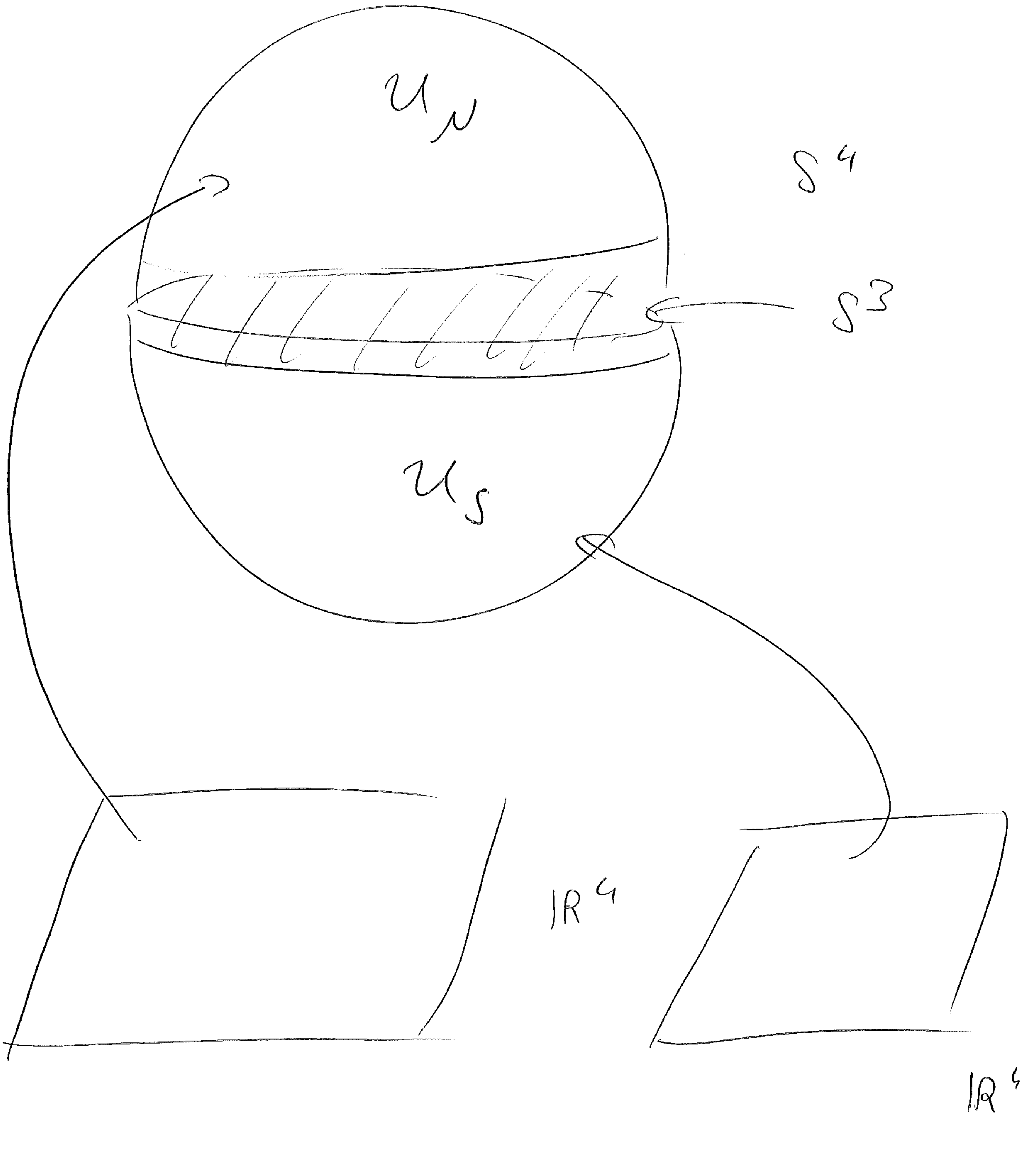
$\mathbb{R}^4 \to \mathcal{U}_S$ and gauge field $A_{S\mu}$

Requirement of smoothness: $\mathrm{tr}\, F_{\mu\nu}^2(A_N) = \mathrm{tr}\, F_{\mu\nu}^2(A_S)$ on $\mathcal{U}_N \cap \mathcal{U}_S$

$$\boxed{A_N = \frac{1}{g} \mathcal{U} D_\mu \mathcal{U}^+} \quad \text{on } \mathcal{U}_N \cap \mathcal{U}_S$$

with transition functions $\mathcal{U}$:

$$\boxed{\mathcal{U} : \mathcal{U}_N \cap \mathcal{U}_S \longrightarrow SU(2)}$$


U_N
S^4
S^3
U_S
$\mathbb{R}^4$
$\mathbb{R}^4$

We are only interested in the top. information stored in the equator

e.g. $$-\int_{\mathcal{U}_S} d^4x \, \mathrm{tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu} = 2\int_{\mathcal{U}_S} d^4x \, \partial_\mu K_\mu(A_S)$$

$$= 2\int_{\partial\mathcal{U}_S \simeq S^3} dS^3_\mu \, K_\mu(A_S)$$

$$= 2\int_{S^3} dS^3_\mu \, K_\mu(\mathcal{U}\, \mathcal{D}_\nu(A_N)\, \mathcal{U}^+)$$

Deform A_N such that $A_N \hat{=} 0$ (or A_S)

$$\Rightarrow -\int_{S^4} d^4x \, \mathrm{tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu} = -\int_{\mathcal{U}_S} d^4x \, \mathrm{tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu}$$

$$= 2\int_{S^3} dS^3_\mu \, K_\mu(\mathcal{U}\, \partial_\rho \mathcal{U}^+)$$

$\simeq$ degree of map

$\Rightarrow$ $\boxed{\text{top. information resides solely in transition fcts. } \mathcal{U}.}$

Example:

$$\mathcal{U}(x) = \frac{1}{\|x\|}\left(x_0\sigma_0 - i\vec{x}\cdot\vec{\sigma}\right)$$

with $\mathcal{U}\mathcal{U}^+ = \frac{1}{x^2}\left(x_0\sigma_0 - i\vec{x}\vec{\sigma}\right)\left(x_0\sigma_0 + i\vec{x}\vec{\sigma}\right)$

$$= \frac{1}{x^2}\left(x_0^2 + \vec{x}^2\right)\mathbb{1} = \mathbb{1}$$

with $(\vec{x}\vec{\sigma})^2 = \frac{1}{2}x_i x_j\left(\{\sigma_i,\sigma_j\} + [\sigma_i,\sigma_j]\right)$

$$= \frac{1}{2}x_i x_j \underbrace{\{\sigma_i,\sigma_j\}}_{2\delta_{ij}} = \vec{x}^2$$

Define $\hat{x} = x_\nu/\|x\| \in S^3$

$\Rightarrow \mathcal{U}(x) \sim S^3 \to S^3$ (is not uniformly well-defined at $x=0$)

- Remark

~ Compute

$$\boxed{q = -\frac{1}{16\pi^2}\int d^4x \, \mathrm{tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu}}$$

$$= \frac{1}{8\pi^2}\int dS^3_\mu \, K_\mu$$

for $A_S = 0$, $A_\nu = \frac{1}{g}\mathcal{U}\partial_\nu\mathcal{U}^+$

Simplification
$A = \mathcal{U}\, d\,\mathcal{U}^+$

$$dA = d\mathcal{U}\, d\mathcal{U}^+ = d\mathcal{U}\,\mathcal{U}^+\mathcal{U}\, d\mathcal{U}^+$$

$$= -\mathcal{U}\, d\mathcal{U}^+\,\mathcal{U}\, d\mathcal{U}^+ = -A^2$$

$$\Rightarrow \quad A\,dA + \frac{2}{3}A^3 = -A^3 + 2/3\, A^3$$

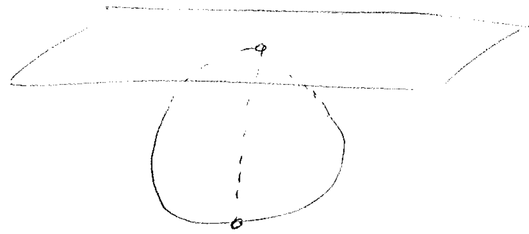
$$= -\frac{1}{3}A^3$$

$$\Rightarrow \boxed{K = +\frac{1}{3}\,\mathrm{tr}\, A^3}$$

$$q = \frac{1}{8\pi^2} \int_{S^3} dS^3_\nu \, K_\nu$$

Evaluation at North pole $(x_0 = 1, \ \vec{x} = 0)$

$$\Rightarrow \vec{A} = i \vec{\sigma}$$



$$K_0 = \frac{1}{3} \underbrace{\varepsilon_{ijk}}_{\varepsilon_{0ijk}} \operatorname{tr} \sigma_i \sigma_j \sigma_k (-i)$$

σ's on page 49

$$= \frac{12}{3} = 4$$

$$\Rightarrow \boxed{q = \frac{1}{2\pi^2} \int dS^3 = 1}$$

Remark:

$$\deg U = \frac{1}{24\pi^2} \int dS^3_\nu \, \varepsilon_{\nu\gamma\rho\sigma} \operatorname{tr} U \partial_\gamma U^+ U \partial_\rho U^+ U \partial_\sigma U$$

$$= -\frac{1}{16\pi^2} \int d^4x \operatorname{tr} F_{\mu\nu} \tilde{F}_{\mu\nu}$$ = Pontryagin index

Instantons : smooth configurations
with top. charge $\boxed{q = 1}$

regular gauge (equator at ∞) $(\partial_\mu A_\mu = 0)$

position of instanton

$$A_\mu^a(x; z, \rho) = 2\,\eta^a_{\mu\nu}\,\frac{(x-z)_\nu}{(x-z)^2 + \rho^2}$$

width

$$= -\,\eta^a_{\mu\nu}\,\partial_\nu \ln \phi(x)$$

with $\phi(x) = \dfrac{1}{(x-z)^2 + \rho^2}$

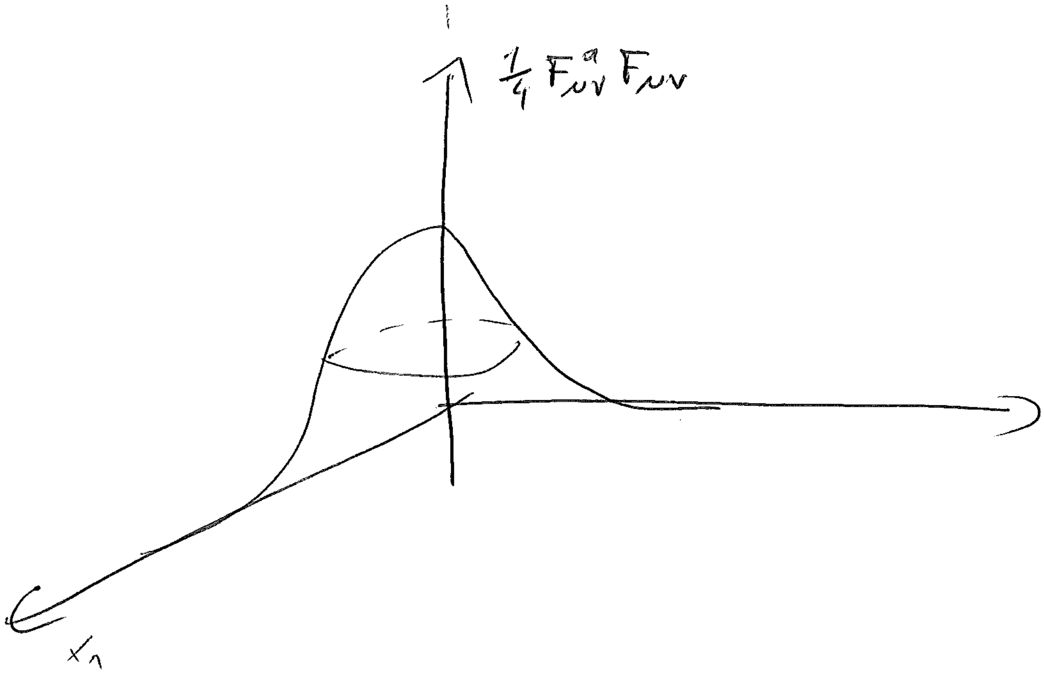
and 't Hooft symbols $\eta^a_{\mu\nu}$

$$\eta^a_{\mu\nu} = -\eta^a_{\nu\mu}\;;\quad \eta^a_{ij} = \varepsilon^{aij}\,,\quad \eta^a_{0i} = -\delta^{ai}$$

z, ρ : moduli parameter : lead to normalisable zero - modes $\Rightarrow$ page action: $(z=0)$

$$F^a_{\mu\nu} = -4\,\eta^a_{\mu\nu}\,\frac{\rho^2}{(\rho^2+x^2)^2}$$

$$\Rightarrow \quad \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} = 4\cdot\underbrace{\eta^a_{\mu\nu}\,\eta^a_{\mu\nu}}_{12}\,\frac{\rho^4}{(\rho^2+x^2)^4}$$



$$S_{YM}[A] = \frac{1}{4g^2}\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu} = 4\cdot 12\; 2\pi^2 \underbrace{\int_0^\infty dx\, x^3\,\frac{\rho^4}{(x^2+\rho^2)^4}}_{1/12}$$

$$= 8\pi^2/g^2$$

Remark: <u>singular gauge</u>

Zeromodes:

ρ:

$$\left.\frac{\delta S_{YM}}{\delta A_\mu}\right|_{A_\mu = A_\mu^a(x_\nu; z,\rho)} = 0$$

$$\overset{\shortparallel}{A_I}$$

$$\Rightarrow \quad \frac{\partial}{\partial \rho}\left(\left.\frac{\delta S_{YM}}{\delta A_\mu^a}\right|_{A_I}\right) = \int d^4 \; \frac{\delta^2 S_{YM}}{\delta A_\mu^a(x)\,\delta A_\nu^b(y)} \cdot \frac{\partial A_\nu^b(y; z,\rho)}{\partial \rho}$$

$$= 0$$

$$\Rightarrow \boxed{\frac{1}{N}\frac{\partial A_I}{\partial \rho} \quad \text{zero mode of fluctuation operator} \quad \frac{\delta^2 S}{\delta A^2}(A_I)}$$

if normalisable

$$\frac{\partial A_{I\,\mu}^a}{\partial \rho} = -4\rho\,\eta^a_{\mu\nu}\frac{(x-z)_\nu}{\left[(x-z)^2+\rho^2\right]^2} = A^a_{\rho\,\mu}$$

is normalisable

Normalisation: Exercise

Normalisation

7.6a

$$\frac{\partial A_\nu^a}{\partial \rho} = -4\,\eta_{\mu\nu}^a \frac{(x-z)_\nu}{\left((x-z)^2+\rho^2\right)^2}\,\rho$$

$$\Rightarrow \int d^4x\, \frac{\partial A_\nu^a}{\partial \rho}\frac{\partial A_\nu^a}{\partial \rho} = 4\rho^2 \eta_{\mu\nu}^a \eta_{\mu\rho}^a \int d^4x\, \frac{x_\nu x_\rho}{(x^2+\rho^2)^4}$$

$$= \frac{4}{4}\eta_{\mu\nu}^a \eta_{\mu\nu}^a \int d^4\hat{x}\, \frac{\hat{x}^2}{(\hat{x}^2+1)^4}$$

$$= \underbrace{\eta_{\mu\nu}^a \eta_{\mu\nu}^a}\, 2\pi^2 \cdot \underbrace{\int_0^\infty d\hat{x}\, \frac{\hat{x}^5}{(\hat{x}^2+1)^4}}_{1/6}$$

$$= \pi^2/3 \cdot 12 = 6\pi^2$$

$$= \pi^2$$

z:

$$\partial_{z_\rho} A^a_\nu(x; z, \rho) = -2\,\eta^a_{\nu\rho} \frac{1}{(x-z)^2+\rho^2}$$

$$+ 4\,\eta^a_{\nu\sigma} \frac{(x-z)_\rho (x-z)_\nu}{\left((x-z)^2+\rho^2\right)^2}$$

evidently not normalisable, but

$$A_{z_\rho} = -\frac{1}{2}\,\partial_{z_\rho} A_\nu(x; z, \rho) + \boxed{D_\nu u(\rho)}$$

imp. for normalisation

with

$$u(\rho) = -\eta^a{}_{\rho\nu}(x-z)_\nu \frac{1}{(x-z)^2+\rho^2}$$

It follows ...:

$$\boxed{A_{z_\rho \nu} = 2\,\eta^a{}_{\nu\rho} \frac{1}{(\rho^2+x^2)^2}}$$

Normalisation: Exercise

Normalisation:

$$A_{z_\rho\,\mu} = 2\,\eta^a_{\mu\rho}\,\frac{1}{(\rho^2+x^2)^2}$$

$$\int d^4x\; A^a_{z_\rho\,\mu}\,A^a_{z_\sigma\,\mu} = 4\,\underbrace{\eta^a_{\mu\rho}\,\eta^a_{\mu\sigma}}_{3\,\delta_{\rho\sigma}}\cdot\int d^4x\,\frac{1}{(\rho^2+x^2)^4}$$

$$= 12\,\delta_{\rho\sigma}\cdot 2\pi^2\underbrace{\int_0^\infty dx\,x^3\,\frac{1}{(\rho^2+x^2)^4}}_{1/12}$$

$$= \underline{2\pi^2\,\delta_{\rho\sigma}}$$

Evidently: $\boxed{\int d^4x\; A^a_{z_\rho\,\mu}\cdot A^a_{\rho\,\mu} = 0}$

$\Rightarrow$ zero modes are orthogonal

Counting of moduli

$$8n - 3$$

$8n$: 4 translations + 1 scaling + 3 (relative) gauge rotations

3: global gauge degrees of freedom

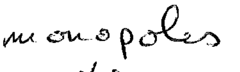
Remarks:

(1) # of moduli depends on space-time manifold $\mathcal{M}$, as does the volume of the moduli space, e.g. $\mathcal{M} = T^4$: $8n$ for $n > 1$

$|n| = 1$: no instanton — v. Baal

Existence $|n| > 1$: Taubes

(2) instantons depends on $\mathcal{M}$

- $\mathbb{R}^4$: 
- $T \times \mathbb{R}^3$:  monopoles for instantons with holonomy
- $T^2 \times \mathbb{R}^3$: monopoles & instanton cores
- $T^3 \times R$: vortices
- T^4: all substructures

Interlude

Importance of zeromodes in QFT:

saddle point expansion:

Generating fct. of YM-theory:

$$Z[J] = \int \mathcal{D}A \; e^{-S_{YM}[A] + \int d^4x \, J^a_\nu A^a_\nu}$$

$$= \sum_{n\in\mathbb{Z}} \int [\mathcal{D}A]_{q(A)=n} \, e^{-S_{YM}[A] + \int d^4x \, J^a_\nu A^a_\nu}$$

(anti-) self dual

$A_\mu = A_{top} + a \leftarrow q(a)=0$ ← fluctuations

$q[A_{top}] = n$

$$= \sum_{n\in\mathbb{Z}} \int \mathcal{D}a \; e^{-S_{YM}[A_{top}+a] + \int d^4x \, J^a_\nu A^a_\nu}$$

$$= \sum_{n\in\mathbb{Z}} \int \mathcal{D}a \; e^{-\Big(\underbrace{S_{YM}[A_{top}]}_{8\pi^2/g^2} + \int d^4x \left.\frac{\delta S}{\delta A^a_\nu}\right|_{A_{top}} \cdot a^a_\nu + \frac{1}{2}\int d^4x\, d^4y \left(\frac{\delta^2 S}{\delta A^2}\right)^{ab}_{\mu\nu,\,A_{top}}(x,y)\, a^a_\mu a^b_\nu + \mathcal{O}(a^3) + \int d^4x \, J^a_\nu A^a_\nu\Big)}$$

$$\simeq \sum_{n\in\mathbb{Z}} e^{-\frac{8\pi^2}{g^2}n} \int \mathcal{D}a \; e^{-\frac{1}{2}\int d^4x\, d^4y\, \frac{\delta^2 S}{\delta A^2}[A_{top}]\cdot a^2} \; e^{\int d^4x \, J\cdot A}$$

(normalisable) zero modes $\downarrow$ fluctuations

$$\simeq \underbrace{\sum_{n\in\mathbb{Z}} e^{-\frac{8\pi^2 n}{g^2}} \int \underbrace{\prod_{i=1}^{N} \mathcal{D}a_{0_i}}_{\sim V_{\text{moduli}}} \int \mathcal{D}a' \; e^{-1/2 \int d^4x\, d^4y \, \frac{\delta^2 S}{\delta A^2}[A_{top}]\cdot a'^2} \cdot e^{\int d^4x \cdot J \cdot a'}}_{Z_n[J]}$$

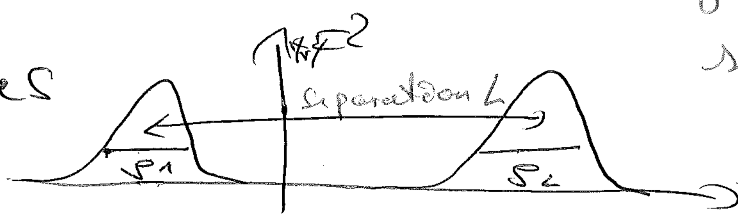
$$\boxed{\begin{aligned} &\simeq \sum_{n\in\mathbb{Z}} e^{-8\pi^2 n/g^2} \int \prod_{i=1}^{N} \mathcal{D}a_{0_i}\, e^{\int d^4x\, J_0 \cdot a_0} \leftarrow \text{zeromode part of } a \\ &\cdot \underbrace{\int \mathcal{D}a' \, e^{-\frac{1}{2}\int d^4x\, d^4y \; a' \cdot \frac{\delta^2 S}{\delta A^2}\cdot a' + \int d^4x\, J' a'}}_{\det'^{-1/2} \frac{\delta^2 S}{\delta A^2}[A_{top}]\; e^{-\int d^4x\, d^4y\, J' \cdot \frac{1}{\delta^2 S/\delta A^2}\cdot J'}} \end{aligned}}$$

dilute gas expansion: $1/g^2 \gg 1$

$$\Rightarrow Z[J] = Z_0[J] + Z_1[J]$$

$$\simeq Z_0[J] + Z_1[J] + \underbrace{\tfrac{1}{2} Z_1[J]^2}_{\text{top charge 2 sector}} + \ldots$$

$\Rightarrow$ like dilute gas



$\boxed{\rho_i/L \ll 1}$

Collective coordinates : $\boxed{q=1}$

$$\prod_{i=1}^{N} \mathcal{D} a_{0i} = \int_0^\infty d\rho \int_{\mathbb{R}^4 \, (S^4)} d^4 z \; J_{\rho, z}$$

$$\mathcal{D} a_{0i} = \prod_i d c_{0i}$$

$J_{\rho,z}$: Jacobian

J factorises into $J_\rho \cdot J_z$

with $c_{0i} = (a_{0i}, A) = \int d^4x \; a_{0i\,\mu}^{a} \, A_\mu^a$

$$\Rightarrow \boxed{J_\rho \sim \int d^4x \cdot A_{\rho\,\mu}^{a} \cdot A_{\rho\,\mu}^{a}}$$