3.3 Fermionic zero modes & Atiyah-Singer Index theorem

$$S[A,\psi,\bar\psi] = S_{YM}[A] + S_D[A,\psi,\bar\psi]$$

with

$$S_D[A,\psi,\bar\psi] = i\int d^4x\, \bar\Psi \underbrace{\gamma_\nu \overset{\partial_\nu + A_\nu}{D_\nu}}_{\not{D}} \psi$$

Question: how many fermionic zero modes do we have in a given gauge field config.?

$$\boxed{\not{D}\, \varphi_{0_i} = 0}$$

zero modes & chirality:

$\gamma_5 = -\gamma_0\gamma_1\gamma_2\gamma_3$

$= -\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$

with $\{\gamma_5, \gamma_\mu\} = 0$

$\gamma_5^2 = 1$

$$i\not{D}\,\varphi_n = \underset{\mathbb{R}}{\lambda_n}\,\varphi_n$$

$$\Rightarrow i\not{D}\,\gamma_5\varphi_n = -\gamma_5 i\not{D}\,\varphi_n$$

$$= -\lambda_n\gamma_5\varphi_n$$

$\Rightarrow$ eigenmodes $\varphi_n, \gamma_5\varphi_n$ are paired with eigenmodes $\pm\lambda_n$

φ_{0_i} and $\gamma_5\varphi_{0_i}$ are zero modes:

diagonalise $\gamma_5, \not{D}$ on zeromode space

$$\boxed{\gamma_5\varphi_{0_i} = \pm\varphi_{0_i}}$$

$$\not{D} = \begin{pmatrix} 0 & \not{D}_- \\ \not{D}_+ & 0 \end{pmatrix}$$

in chiral repr.

Furthermore: chiral transformations

$$\psi \to e^{+i\alpha\gamma_5}\psi = \psi^\alpha$$

$$\bar\psi \to \bar\psi e^{+i\alpha\gamma_5} = \bar\psi^\alpha$$

Symmetry of $S_D[A,\psi,\bar\psi]$ as

$$\bar\psi\not{D}\psi \to \bar\psi e^{i\alpha\gamma_5}\not{D}e^{i\alpha\gamma_5}\psi = \bar\psi\,\overbrace{e^{i\alpha\gamma_5}e^{-i\alpha\gamma_5}}^{\mathbb{1}}\not{D}\psi$$

Noether current:

$$\partial_\nu j_\nu^5 = \frac{\delta S_D}{\delta\alpha} = \partial_\nu \underbrace{\left(\bar\psi \gamma_\nu \gamma_5 \psi\right)}_{j_\nu^5} = 0$$

What happens at the quantum level:

$$Z_D[\eta, \bar\eta ; A] = \int \mathcal{D}\psi \mathcal{D}\bar\psi \, e^{-S_D[A,\psi,\bar\psi]} \cdot e^{\int d^4x \left(\bar\eta \psi - \bar\psi \eta\right)}$$

$$= \int \mathcal{D}\psi^\alpha \mathcal{D}\bar\psi^\alpha \, e^{-S_D[A,\psi^\alpha,\bar\psi^\alpha]} \, e^{\int d^4x \left(\bar\eta \psi^\alpha - \bar\psi^\alpha \eta\right)}$$

for now $\eta, \bar\eta = 0$: $\frac{\delta Z_D}{\delta\alpha} = 0$:

$$\text{infinitesimal} = \int \mathcal{D}\psi \mathcal{D}\bar\psi \left(\frac{\delta J}{\delta\alpha} - \partial_\nu\left(\bar\psi \gamma_\nu \gamma_5 \psi\right)\right) e^{-S_D}$$

with

$$J = \frac{\partial(\psi^\alpha, \bar\psi^\alpha)}{\partial(\psi, \bar\psi)} \qquad \text{Jacobian}$$

Naively: chiral transformation

$$\psi \to \psi^{\alpha} = e^{+i\gamma_5 \alpha}\psi \quad \text{is (chiral) rotation}$$

with $\det J = 1$.

Computation of J: $\psi = \sum_n a_n \varphi_n$

$$\mathcal{D}\psi^{(\alpha)}\mathcal{D}\bar{\psi}^{(\alpha)} \simeq \prod_n da_n^{(\alpha)}\, d\bar{b}_n^{+(\alpha)}$$

Grassmann variables

with $\psi^{(\alpha)} \simeq (1+i\alpha\gamma_5)\psi$

and hence $a_n^{(\alpha)} = (\varphi_n, \psi^{(\alpha)}) = \sum_m \underbrace{\left(\delta_{nm} + i\alpha(\varphi_n, \gamma_5\varphi_m)\right)}_{T_{nm}} a_m$

similarly $\bar{b}_n^{(\alpha)} = (\bar{\psi}^{(\alpha)}, \varphi_n) = \sum_m \left(\delta_{nm} + i\alpha(\varphi_m, \gamma_5\varphi_n)\right)\bar{b}_m$

Remark: $\int da = 0$, $\int da\, a = 1$ for Grassmann variable a

$$\Rightarrow \int d(\gamma a)(\gamma a) = 1, \quad \gamma \in \mathbb{C}$$

$$\Rightarrow \boxed{d(\gamma a) = \gamma^{-1} da}$$

It follows,

$$\mathcal{J} = \left(\det^{-1} T\right)^2$$

and $\ln \mathcal{J} = -2 \ln \det T = -2 \operatorname{Tr} \ln T = 2 \sum_n \ln T_{nn}$

$\partial_\mu \alpha = 0 \Rightarrow$ $= -2i\alpha \sum_n \left(\varphi_n, \gamma_5 \varphi_n\right) + \mathcal{O}(\alpha^2)$

$$\Rightarrow \boxed{\ln \mathcal{J} = -2i\alpha \operatorname{Tr} \gamma_5 + \mathcal{O}(\alpha^2)}$$

$\gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$: Naively $\operatorname{Tr} \gamma_5 = 0$

What about the zero modes:

$$\ln \mathcal{J} = -2i\alpha \sum_n \left(\varphi_{0n}, \gamma_5 \varphi_{0n}\right)$$

$$= -2i\alpha\left(\nu_+ - \nu_-\right) = -2i\alpha \operatorname{index} \nabla\!\!\!\!/_+$$

with

$$\gamma_5 \varphi_{0n} = \pm \varphi_{0n}, \quad D\!\!\!\!/\, \varphi_{0n} = 0$$

ν_+ : # of zero modes with positive chirality

ν_- : # of zero modes with negative chirality

Computation of $\ln J(A)$:

now α-dep.

$$-\frac{1}{2i}\ln J(A) = \operatorname{Tr}\,\alpha\,\gamma_5 := \lim_{\varepsilon\to 0} \operatorname{Tr}\,\alpha\,\gamma_5\, e^{\varepsilon \not{D}^2}$$

$$= \lim_{\varepsilon\to 0}\sum_n (\varphi_n, \alpha\,\gamma_5\,\varphi_n)\, e^{-\lambda_n^2}$$

$$\sum_n (\varphi_n, \alpha\gamma_5 e^{\varepsilon\not{D}^2}\varphi_n) = \int d^4x\,\alpha(x) \underbrace{\sum_n (\varphi_n, x)\,\alpha\gamma_5\, e^{\varepsilon(\not{\partial}+\not{A})^2}\cdot (x,\varphi_n)}_{\not{\delta}\ (\text{for } \varepsilon\to 0)}$$

$$= \int d^4x\,\alpha(x)\int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}(2\pi)^4\,\delta(p-p')$$

$$\cdot \operatorname{tr}\gamma_5\, e^{\varepsilon(i\not{p}+\not{\partial}+\not{A}(x))^2}$$

We use

$$(\not{p}+\not{\partial}+\not{A})^2 = \frac{1}{2}\Big(\underbrace{\{\gamma_\mu,\gamma_\nu\}}_{2\delta_{\mu\nu}} + \underbrace{[\gamma_\mu,\gamma_\nu]}_{\sigma_{\mu\nu}}\Big)\, \overset{\partial_\mu + p_\mu + A_\mu}{\overset{\|}{D_{p\mu}}}\, D_{p\nu}$$

$$\Rightarrow \boxed{\not{D}_p^{\,2} = D_p^2 + \frac{1}{4}\sigma_{\mu\nu}\underset{[D_{p\mu},\, D_{p,\nu}]}{\underset{\|}{F_{\mu\nu}}}}$$

$$\Rightarrow \mathcal{A} = \lim_{\varepsilon\to 0}\int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\, \gamma_5\, e^{-\varepsilon\left(-p^2 + \frac{1}{4}\sigma_{\mu\nu}F_{\mu\nu}\right)}$$

$$+\, \mathcal{O}(\varepsilon)$$

$$= \lim_{\varepsilon\to 0}\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\frac{1}{16}\,\mathrm{tr}\left(\gamma_5\,\sigma_{\mu\nu}\,\sigma_{\rho\sigma}\right) F_{\mu\nu}F_{\rho\sigma}\, \varepsilon^2 e^{-\underbrace{\varepsilon p^2}_{\hat{p}^2}}$$

$$= \frac{1}{2}\frac{1}{16}\left[\mathrm{tr}\left(\gamma_5\,\sigma_{\mu\nu}\,\sigma_{\rho\sigma}\right) F_{\mu\nu}F_{\rho\sigma}\right]\left(\int\frac{d^4\hat{p}}{(2\pi)^4}\, e^{-\hat{p}^2}\right)$$

---

(1) $$\mathrm{tr}\, \gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = -4\,\varepsilon_{\mu\nu\rho\sigma}$$

$$\Rightarrow \mathrm{tr}\, \gamma_5\,\sigma_{\mu\nu}\,\sigma_{\rho\sigma} = -16\,\varepsilon_{\mu\nu\rho\sigma}$$

(2) $$\int\frac{d^4\hat{p}}{(2\pi)^4}\, e^{-\hat{p}^2} = \frac{1}{16\pi^2}$$

$$\Rightarrow \boxed{\mathcal{A}(x) = -\frac{1}{32\pi^2}\, \varepsilon_{\mu\nu\rho\sigma}\, \mathrm{tr}\, F_{\mu\nu} F_{\rho\sigma}}$$ covariant anomaly

In summary: $J = e^{-2i \int d^4x\, \alpha(x)\, \mathcal{A}(x)}$

(1) locally:

$$\boxed{\partial_\mu j^5_\mu(x) = -2i\, \mathcal{A}(x)}$$

(2) globally:

$$\underbrace{-\frac{1}{32\pi^2}\, \varepsilon_{\mu\nu\rho\sigma} \int d^4x\, \mathrm{tr}\, F_{\mu\nu} F_{\rho\sigma}}_{} = \nu_+ - \nu_-$$

p. 71 $$\boxed{q = \nu_+ - \nu_-}$$

Atiyah-Singer index theorem

(for a twisted spinor complex in a trivial background geometry) lhs quick & dirty

introduce mass : $\bar{\Psi}(i\not{D} + m)\Psi$

$$\Rightarrow \quad \langle \partial_\nu \bar{\Psi} \gamma_\nu \gamma_5 \Psi \rangle + 2im \langle \bar{\Psi} \gamma_5 \Psi \rangle = -2i\mathcal{A}$$

(1) $\int_{\mathbb{R}^4} \partial_\nu \langle \bar{\Psi} \gamma_\nu \gamma_5 \Psi \rangle = 0$

as fermions are massive $\Rightarrow \int_{\partial\mathbb{R}^4} \langle \bar{\Psi} \gamma_\nu \gamma_5 \Psi \rangle d\bar{F}_\nu = 0$

(2) $2im \int d^4x \langle \bar{\Psi}(x) \gamma_5 \Psi(x) \rangle$

$= -2im \int d^4x \operatorname{tr} \gamma_5 \langle \Psi(x) \bar{\Psi}(x) \rangle$

$= -2im \operatorname{Tr} \gamma_5 \langle \Psi(x) \bar{\Psi}(x) \rangle$

Propagator

$= -2im \operatorname{Tr} \gamma_5 \dfrac{1}{i\not{D} + m}$

$= -2im \sum_{n} \left(\varphi_n, \gamma_5 \dfrac{1}{\lambda_n + m} \varphi_n \right)$

$= -2im \sum_{n_0} \dfrac{1}{m} \left(\varphi_{n_0}, \gamma_5 \varphi_{n_0} \right) = -2i(\nu_+ - \nu_-)$

$$\Rightarrow \boxed{\int \mathcal{A}\, d^4x = \nu_+ - \nu_-}$$

Interlude II: a glimpse at chiral symmetry breaking

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S_D[A,\psi,\bar{\psi}] + \int d^4x (\bar{\eta}\psi - \bar{\psi}\eta)}$$

$\psi', \bar{\psi}'$: non-zero modes: $(\varphi_{n_0}, \psi') = 0$

$$= \int \mathcal{D}\psi' \mathcal{D}\bar{\psi}'\, e^{-S_D[A,\psi',\bar{\psi}'] + \int d^4x (\bar{\eta}\psi - \bar{\psi}\eta)}$$

$$\cdot \prod_n \int da_{0n}\, d\bar{b}_{0n}\, e^{-m\int d^4x\, \bar{\psi}_0\psi_0 + \int d^4x (\bar{\eta}\psi_0 - \bar{\psi}_0\eta)}$$

$$\simeq \underset{\substack{\uparrow \\ \text{det in non-zero mode space}}}{\det{}'}(i\not{D} + m)\, e^{-\int d^4x\, \bar{\eta}' \frac{1}{i\not{D}+m}\eta}$$

$$\cdot \prod_n \int da_{0n} d\bar{b}_{0n}\, e^{-m\sum_n \bar{b}_{0n} a_{0n} + \sum_n \int d^4x (\bar{\eta}\varphi_{0n} a_{0n} - \bar{b}_{0n}\varphi_{0n}^{\dagger}\eta)}$$

$$= \det{}'(i\not{D} + m)\, e^{-\int d^4x\, \bar{\eta}' \frac{1}{i\not{D}+m}\eta}$$

$$\underbrace{\frac{(-1)^N}{N!} \prod_{n=1}^{N} \left[m - \int d^4x\, \bar{\eta}\varphi_{0n} \int d^4x'\, \varphi_{0n}^{\dagger}\eta \right]}_{Z_{0_N}}$$

Fermionic zero mode: $(\varphi_0, \varphi_0) = 1$

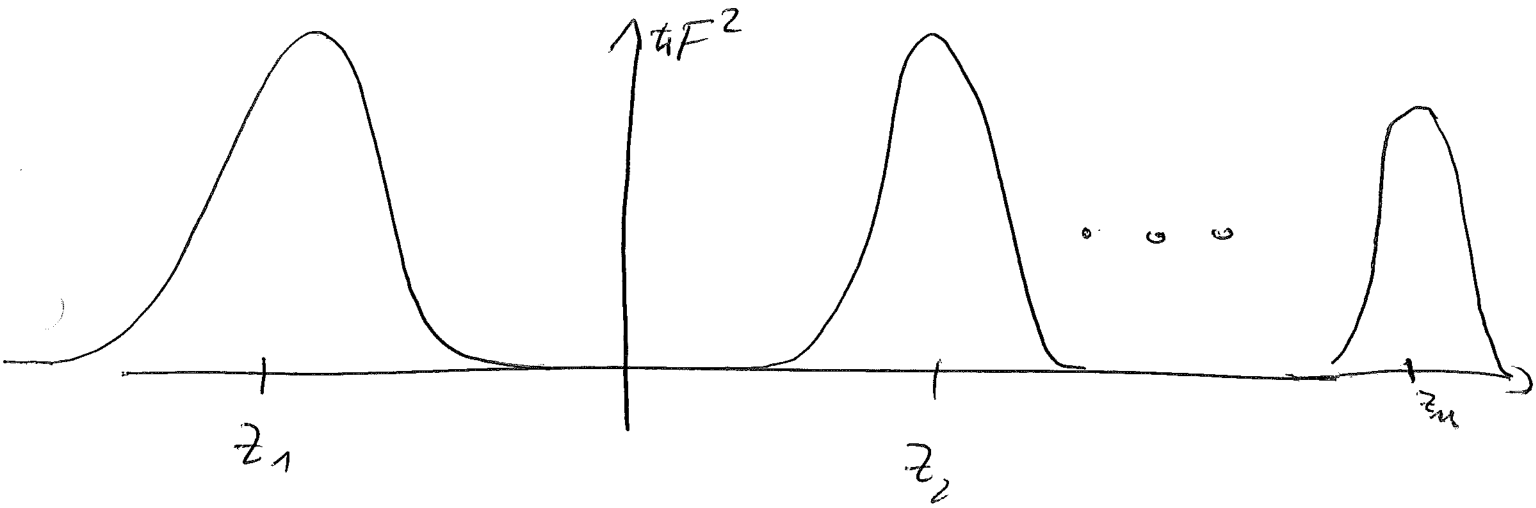
for instantons: $q=1$

$$\varphi_0^a(x; z, \rho) = \frac{\sqrt{2}}{\pi} \frac{\rho}{\left((x-z)^2 + \rho^2\right)^{3/2}} u^a$$

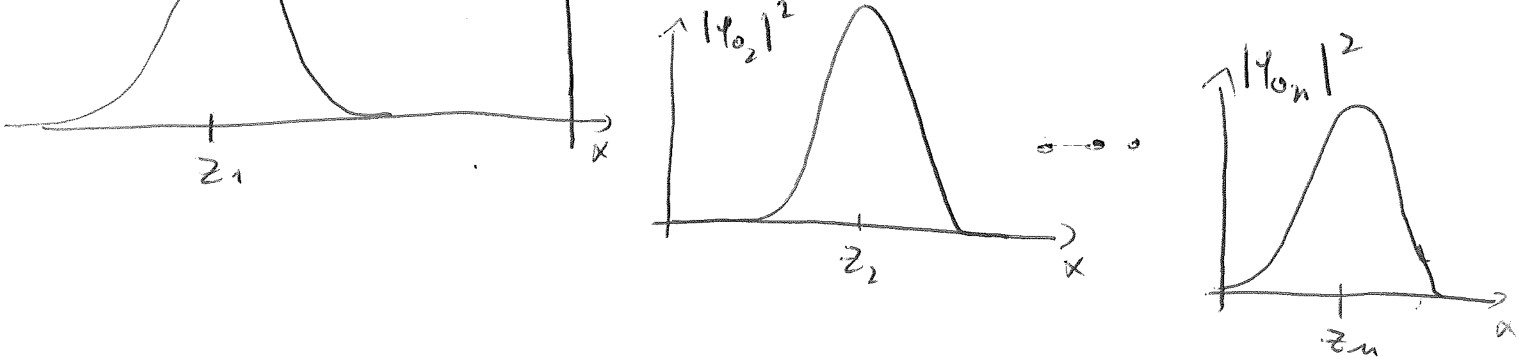
with

$$\sum_a u^a \times u^{+a} = \frac{1+\gamma_5}{2}$$

$\Rightarrow$ Assume lump-like zero modes for multi-instantons:



zero-modes: for $\boxed{\rho_n \to 0}$ ← extension of $\eta, \bar{\eta}$



$$\Rightarrow Z_{0_N} \simeq \frac{1}{N!}\left[(-m)^N + \ldots + C_N(\rho)\prod_{n=1}^{N} \bar\eta(z_n)\,\frac{1+\gamma_5}{2}\,\eta(z_n)\right]$$

with
$$c_N(\rho) \sim \left(\rho\int d^4x\left(\frac{1}{(x^2+\rho^2)^{3/2}} - \left(\frac{1}{x^2}\right)^{3/2}\right)\cdot\frac{\sqrt{2}}{\pi}\right)^2$$

'normalisation'

$$= -2^5\,\pi^2\rho^4$$

+ Gauge field quant.:

$$Z_{0_N} = \left[\prod_n \int d^4z_n\, d\rho_n\, \nu(\rho_n)\right]\cdot Z_{0_N}\cdot \underbrace{\overbrace{\frac{\det'(i\not{D}+m)}{\det(i\not{\partial}+m)}}^{\text{normalis.}}}_{\sim\rho}$$

$$\sum_{N\in\mathbb{Z}} Z_{0_N} \simeq e^{Z_{0_1}+Z_{0_{-1}}}$$

with

$$Z_{0_{\pm1}} = \int d^4z \int d\rho\,\nu(\rho)\,c_1(\rho)\;\bar\eta(z)\,\frac{1\pm\gamma_5}{2}\,\eta(z)$$

$$\Rightarrow \boxed{Z_{0_1}+Z_{0_{-1}} \simeq \int d^4z\int d\rho\,\rho\,\nu(\rho)\,c_1(\rho)\;\bar\eta\cdot\eta}$$

$\Rightarrow$ (addit.) anomalous mass-term for fermions

Remarks:

(i) $\mathcal{N}(\rho)\, c_1(\rho) \sim e^{-8\pi^2/g^2(\rho)} \cdot \rho^{5}\ \rho^{-5}$

Fermions

$N_f{+}1$

ρ^{5N_f}

gauge fields

(ii) $Z_{\mathcal{O}_1} + Z_{\mathcal{O}_{-1}} \longrightarrow \Delta \int d^4z\, \bar{\psi}\psi$

$$\Delta \sim \int d\rho\, \mathcal{N}(\rho)\, c_1(\rho)\,/\rho^2$$

(iii) More flavours, i.e. $N_f = 3$: $(N_f = 2+1)$

$$\Delta \int d^4z\, \bar{\psi}\psi \to \Delta \sum_{\pm} \int d^4z \det_{s,t} \bar{\psi}_s \frac{1\pm\gamma_5}{2}\psi_t$$

$$\Delta \sim \int d\rho\, \rho^{3N_f-5}\, e^{-8\pi^2/g^2(\rho)}\left[e^{-M^2\rho^2 c}\right]$$

M: e.g. Higgs mass

$\Rightarrow\ \eta'$ - mass

(iv) Quantitative picture:

(a) instanton-liquid: 'adjust density $\mathcal{N}(\rho)\, c_1(\rho)$'

(b) non-perturbative QFT