3.5 Monopoles & dyons

Generalisation of Abelian Higgs model

chapt. 2.1, p. 36 – 40

$U(1) \longrightarrow SU(2)$ or gauge group G

action: (see also p. 65) Minkowski!

$\eta_{\mu\nu} = (1,-1,-1,-1)$

$$S[\phi, A] = -\frac{1}{2}\int d^4x \, \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\int d^4x \,(D_\mu\phi)^a (D^\mu\phi)^a$$

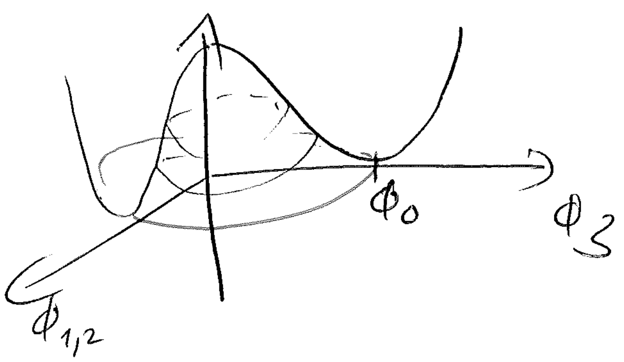
$$+ \int d^4x \, V[\phi]$$

ϕ^a : in adjoint representation of $SU(2)$

$g\phi = U(g)\,\phi\, U^\dagger(g)$ $\simeq SO(3)$

Higgs phase : $\left.\frac{\partial V}{\partial \phi}\right|_{\phi_0} = 0$, $\left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi_0} > 0$, $V[\phi_0] := 0$

e.g. $V = \frac{1}{4}\lambda(\phi^a\phi^a - v^2)^2$



residual gauge symmetry:

little group, stabiliser H (analysis of Lorentz group, def. of particles)

$$\boxed{H = \{ h \in SU(2) \mid h\,\Phi_0 = \Phi_0 \}}$$

and $SU(2)/H \simeq \{ \phi \mid V[\phi] = 0 \} \simeq S^2$

$H = U(1) \times \mathbb{Z}_2 \longleftarrow$ center of $SU(2)$

static solution with finite energy:

$$\mathcal{M}_{\text{space-time}} \simeq \mathbb{R} \times \underbrace{\mathbb{R}^3}_{\mathcal{M}_{\text{space}}}$$

$$\partial \mathcal{M}_{\text{space}} = S^2$$

Classification: $\partial \mathcal{M}_{\text{space}} \to SU(2)/H \simeq S^2$

$$\Rightarrow \boxed{\pi_2(S^2) \simeq \mathbb{Z}}$$

't Hooft Polyakov monopole:

$$E = \frac{1}{2}\int d^3x \left(\vec{E}^{a\,2} + \vec{B}^{a\,2} + (\mathcal{D}_0\phi)^{a\,2} + (\mathcal{D}_i\phi)^{a\,2} + V[\phi] \right)$$

with $E^a{}_i = -F^a{}_{0i}$, $B^a{}_i = -\frac{1}{2}\varepsilon_{ijk}F^a{}_{jk}$

EoM:

$$\boxed{(\mathcal{D}_\mu F^{\mu\nu})^a = -g\,\varepsilon^{abc}\phi^b(\mathcal{D}^\nu\phi)^c}$$

$$\boxed{\mathcal{D}^\mu\mathcal{D}_\mu\phi^a = -\lambda\,\phi^a(\phi^{a2}-v^2)}$$

$|\vec{x}| \to \infty$: $\phi \to v\cdot\hat{\phi}_0$

$$\frac{1}{v}\mathcal{D}_\mu\phi = \partial_\mu\hat{\phi}_0^a + g\,A_\mu^b\,\hat{\phi}_0^c\,\varepsilon^{abc} \to 0$$

$(A_\mu \times \hat{\phi}_0)^a$

$$\rightsquigarrow \quad A_\mu^a = \underbrace{-\frac{1}{g}\varepsilon^{abc}\hat{\phi}_0^b\,\partial_\mu\hat{\phi}_0^c}_{-\frac{1}{g}(A_\mu\times\hat{\phi}_0)^a = -\partial_\mu\hat{\phi}_0} + \hat{\phi}_0^a\,\mathcal{A}_\mu$$

$\frac{1}{v}\phi_0^a$

Abelian gauge field

$\hat{\phi}_0\ \mathcal{A}_\mu \times \hat{\phi}_0 = 0$

$$\boxed{\mathcal{A}_\mu:\ \text{Abelian gauge field} \parallel \text{to vacuum vector } \phi_0}$$

Field strength of unbroken sub group $U(1)$:

$$\boxed{\begin{aligned}\mathcal{F}_{\mu\nu} &= \hat{\phi}_0^a F^a_{\mu\nu} = \hat{\phi}_0^a \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\varepsilon^{abc} A^b_\mu A^c_\nu\right)\\ \hat{\phi}_0^a \partial_\mu \hat{\phi}_0^a = \tfrac{1}{2}\partial_\mu \hat{\phi}_0^{a\,2} = 0 \rightarrow \quad &= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{g}\hat{\phi}_0^a \varepsilon^{abc} \partial_\mu \hat{\phi}_0^b \partial_\nu \hat{\phi}_0^c\end{aligned}}$$

$\mathcal{F}_{\mu\nu}$ is conserved:

$$\begin{aligned}\partial^\mu \mathcal{F}_{\mu\nu} &= \left(\partial^\mu \hat{\phi}_0^a\right) F^a_{\mu\nu} + \hat{\phi}_0^a \partial_\mu F^a_{\mu\nu}\\ &= \left(\partial^\mu \hat{\phi}_0^a\right) F^a_{\mu\nu} - \hat{\phi}_0^a g\,\varepsilon^{abc}\Big(A^b_\mu F^a_{\mu\nu}\\ &\qquad + \underbrace{\hat{\phi}_0^b \left(\mathcal{D}^\mu \hat{\phi}_0\right)^c v^2}_{\hat{\phi}_0^a \varepsilon^{abc}\cdot(\;)=0}\Big)\\ &= \left(\partial^\mu \hat{\phi}_0^a\right) F^a_{\mu\nu} - \underbrace{\hat{\phi}_0^a g\,\varepsilon^{abc} A^b_\mu}_{\partial_\mu \hat{\phi}_0^a} F^a_{\mu\nu}\\ &= 0\end{aligned}$$

102

$$\partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu \hat{\phi}_0^a \cdot \tilde{F}^{\mu\nu a} + \hat{\phi}_0^a \underbrace{\partial_\mu \tilde{F}^{\mu\nu a}}_{\underbrace{-g\,\varepsilon^{abc} A_\mu^b \tilde{F}^{\mu\nu c}}_{-\partial_\mu \phi_0^a \tilde{F}^{\mu\nu a}}}$$

$$\doteq 0$$

$$\boxed{F^{\mu\nu} \overset{\text{(locally)}}{=} \partial_\mu \bar{A}_\nu + \partial_\nu \bar{A}_\mu}$$

) - charges: $\vec{E} = \hat{\phi}_0^a \vec{E}^a$, $\vec{B} = \hat{\phi}_0^a \vec{B}^a$

ric charge:

$$q = \int_{\partial M_{space} \simeq S^2} \vec{E} \cdot d\vec{S} = + \int_{S^2_\infty} \hat{\phi}_0^a E_i^a \, dS^i$$

$$= -\frac{1}{g} \int_{S^2_\infty} \varepsilon^{abc} \hat{\phi}_0^a \, \partial_0 \hat{\phi}_0^b \, \partial_i \hat{\phi}_0^c \, dS^i$$

magnetic charge:

$$m = \int_{S^2_\infty} \vec{B} \cdot d\vec{S} = \int_{S^2_\infty} \hat{\phi}_0^a \vec{B}^a_i \, dS^i$$

$$= -\frac{1}{2g} \int \varepsilon^{abc} \varepsilon^{ijk} \hat{\phi}_0^a \partial_j \hat{\phi}_0^b \partial_k \hat{\phi}_0^c dS^i$$

$\frac{4\pi}{g}$ degree of map $S^2 \to S^2$

$$= -\frac{4\pi n}{g}$$

$$\Rightarrow \boxed{m = -\frac{4\pi n}{g}} \Leftrightarrow$$

$$n = \frac{1}{8\pi} \int d^3x \, J^\nu_{top}$$

with $J^\nu_{top} = \varepsilon^{\nu\rho\sigma\tau} \varepsilon^{abc} \cdot \partial_\rho \phi^a \partial_\sigma \phi^b \partial_\tau \phi^c$

Reformulate

$$q = \int d^3x \; E^{ia} \left(D^i \hat{\phi}_0\right)^a$$

$$m = \int d^3x \; B^{ia} \left(D^i \hat{\phi}_0\right)^a$$

$$q = \int_{S^2_\infty} \hat{\phi}_0^a \, E_i^a \, dS^i = \int_{\mathbb{R}^3} \partial_i \left(\hat{\phi}_0^a E_i^a \right) d^3x$$

$$= \int_{\mathbb{R}^3} \left[\left(\partial_i \hat{\phi}_0^a \right) E_i^a + \hat{\phi}_0^a \underbrace{\partial_i E_i^a}_{-g A_i^b E_i^c \varepsilon^{abc} + \cdots} \right] d^3x$$

$\hat{\phi}_0^a \cdot \ldots = 0$

$$= \int_{\mathbb{R}^3} (D_i \phi)^a E^a{}_i \, d^3x$$

$$m = \int_{S^2_\infty} \hat{\phi}_0^a \, \vec{B}_i^a \, dS^i = \int_{\mathbb{R}^3} \partial_i \left(\hat{\phi}_0^a B_i^a \right) d^3x$$

$$= \int_{\mathbb{R}^3} (D_i \hat{\phi})^a B^a{}_i \, d^3x$$

Bogomol'nyi bound

p. 100

$$E = \frac{1}{2}\int d^3x \left(\vec{E}^{a\,2} + \vec{B}^{a\,2} + \underbrace{(D_\nu \phi)^{a\,2}}_{\sin^2\theta + \cos^2\theta} \right) + \int d^3x\, V[\phi]$$

$$\geq \frac{1}{2}\int d^3x \Big[\left(E - \sin\theta (D\phi) \right)_i^{a\,2} + \left(B - \cos\theta (D\phi) \right)_i^{a\,2}$$

$$+ \underbrace{2 E^a (D\phi)^a \cdot \sin\theta}_{q \cdot v \;\leftarrow\; \frac{1}{2}\int d^3x} + \underbrace{2 B_i^a (D_i\phi)^a \cos\theta}_{m \cdot v \;\leftarrow\; \frac{1}{2}\int d^3x} \Big]$$

Rest mass $M = E$:

$$\Rightarrow \quad M \geq v\,(q \sin\theta + m\cos\theta)$$

bound maximal for $\boxed{\tan\theta = q/m}$

$$\Rightarrow \quad \boxed{M \geq \sqrt{m^2 + q^2}}$$

Bogomol'nyi bound

end lecture

Θ - term : action on p. 98

$$S[\phi, A] = -\frac{1}{2}\int d^4x \,\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\int d^4x (\mathcal{D}_\mu\phi)^a(\mathcal{D}^\mu\phi)^a$$

$$- \int d^4x \, V[\phi] + \frac{\Theta g^2}{16\pi^2}\int d^4x \,\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$$

$$= \int d^4x \, \mathcal{L}[\phi, A]$$

Θ -term breaks CP - invariance (pseudo-scalar)

$$\overset{P}{(q, m) \to (-q, m)}$$

Electric charge related to unbroken $U(1)$: $\hat{H}$ stabiliser

see p. 89

$$\underset{h}{\overset{H}{\Downarrow}} : A_\mu^h = A_\mu \underbrace{- \frac{1}{g}\mathcal{D}_\mu \hat{\phi}}_{\delta_h A_\mu}$$

$$\phi^h = \phi \underbrace{- [\phi, \hat{\phi}]}_{\delta_h \phi} = 0$$

Noether charge N:

$$N = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \partial_0 A_i}\delta_h A_i + \underbrace{\frac{\partial \mathcal{L}}{\partial \partial_0 \phi}\delta_h \phi}_{0} \right)$$

$$= \frac{1}{g}\int d^3x (\mathcal{D}_i \hat{\phi})^a E_i^a - \frac{\Theta g}{8\pi^2}\int d^3x \, B_i^a (\mathcal{D}_i \hat{\phi})^a$$

$$\Rightarrow \boxed{N = \frac{1}{g} q - \frac{\Theta g}{8\pi^2} m} \quad \text{quantised}$$

or :

$$q = g \underset{\underset{\mathbb{Z}}{\Uparrow}}{N} + \Theta \, g^2/8\pi^2 \cdot \underset{\underset{4\pi/g\,\mathbb{Z}}{\Uparrow}}{m}$$

't Hooft - Polyakov monopole: $N = 1$, $m = -4\pi/g$

$$\Rightarrow \quad q = g\left(1 - \Theta/2\pi\right)$$

general Dyon: $\quad m = 4\pi/g \; \underset{\underset{\mathbb{Z}}{\Uparrow}}{n_m}$

$$\Rightarrow \quad q = N g + \frac{\Theta g}{2\pi} n_m$$

or

$$q + i m = g\left(n_q + n_m \tau\right)$$

where $\quad \tau = \dfrac{\Theta}{2\pi} + 4\pi i/g^2$

Bogomol'nyi bound

$$\boxed{M \geq \left| v\, g\left(n_q + n_m \tau\right)\right|}$$

't Hooft - Polyakov solution:

bound saturated for

$$\boxed{\begin{aligned} D_0 \Phi^a &= 0 \\ E_i^a &= (D_i \phi)^a \sin\Theta \\ B_i^a &= (D_i \phi)^a \cos\Theta \end{aligned}}$$

BPS - states

't Hooft - Polyakov monopole: $q = 0$, $m = 1$

$$\boxed{\begin{aligned} \phi^a &= v \frac{x^a}{|x|} H(v g |x|) \\ A_i^a &= \varepsilon^{aij} \frac{x^j}{g \vec{x}^2} \left(1 - K(a g |\vec{x}|)\right) \\ A_0^a &= 0 \end{aligned}} \quad (*)$$

with

$H(y \to \infty) = 1$, $\quad H(y \to 0) = 0$

$K(y \to \infty) = 0$, $\quad K(y \to 0) = 1$

continuity [equator at ∞]

solution:

$$\boxed{\begin{aligned} H(\cancel{v}\, g|x|) &= \coth(v g|x|) - \frac{1}{v g|x|} \\ K(v g r) &= \frac{\sinh(v g|x|)}{v g|x|} \end{aligned}}$$

$$\phi^a(|x| \to \infty) = v \frac{x^a}{|x|} \Rightarrow \hat{\phi}_0^a = \frac{x^a}{|x|}$$

$$m = -\frac{1}{8\pi} \int_{S^2} dS^i \, \varepsilon^{abc} \varepsilon^{ijk} \frac{x^a}{|x|} \partial_j \frac{x^b}{|x|} \partial_k \frac{x^c}{|x|}$$

$$= 1$$

Dyons: $(q \neq 0)$

$$\boxed{A_0^a = \frac{x^a}{\vec{x}^2} J(v g|x|) \quad + \quad (*)}$$