4.3 Low energy effective action

General $N=2$ Lagrangian

$$\mathcal{L} = \frac{1}{4\pi} \operatorname{Im}\left(\int d^2\theta \, \mathcal{F}_{ab}(\phi) W^{a\,\alpha} W^b_\alpha \right.$$

$$\left. + 2 \int d^2\theta \, d^2\bar{\theta} \left(\phi^\dagger e^{2V} \mathcal{F}_a(\phi) \right) \right) \qquad (*)$$

with prepotential $\mathcal{F}(\phi)$:

$$\mathcal{F}_a(\phi) = \frac{\partial \mathcal{F}}{\partial \phi^a}, \quad \mathcal{F}_{ab} = \frac{\partial^2 \mathcal{F}}{\partial \phi^a \partial \phi^b}$$

Low energy effective action: (Wilsonian effect. action)

$$e^{i\mathcal{L}_{eff}[\phi_{IR}]} = \int d\varphi_{UV} \, e^{-iS[\varphi_{UV} + \phi_{IR}]}$$

Construction:

- gauge invariance
- meromorphic $\Leftarrow$ non-renormalisation
- susy

(no explicit way known

Low-energy $\Leftrightarrow$ Derivative expansion

$$\mathcal{L}_{eff} = (*)$$

$\Rightarrow$ one function has to be determined

$$\boxed{\tau(\phi) = \mathcal{F}_{ab}(\phi)}$$

In $N=2$ superfield language:

$$\Phi = \phi + \Theta^{2\,\alpha} W_\alpha + \Theta^{2\,\alpha} \Theta^2_\alpha D^2 \phi$$

$$\left(\phi = \Phi|_{\Theta^2=0} \,,\; W_\alpha = D^2_\alpha \Phi|_{\Theta^2=0} \right)$$

with

$$D^i_{\dot\alpha} \Phi = 0 \,, \quad D^{i\alpha} D^j_\alpha \Phi = \bar{D}^i_{\dot\alpha} \bar{D}^{j\dot\alpha} \Phi$$

$$\mathcal{L}_{eff} = \frac{1}{4\pi} \operatorname{Im} \int d^4x\, d^2\Theta_1 d^2\Theta_2\, \mathcal{F}(\Phi)$$

$$= \frac{1}{8\pi} \operatorname{Im} \left(\int d^2\Theta\, d^2\bar\Theta \left(\underbrace{\Phi_D}_{\frac{\partial \mathcal{F}}{\partial \phi}(\phi)} \phi - \bar\Phi_D \phi \right) + \int d^2\Theta\, \tau(\phi) W^\alpha W_\alpha \right)$$

Perturbation theory:

Non-renormalisation $\Rightarrow$ 1-Loop exact

in holomorphic scheme

$$\Rightarrow \boxed{\mathcal{F}_{1\text{-Loop}}(A) = \frac{i}{2\pi} A^2 \ln A^2/\Lambda^2}$$

"Λ^2_{QCD}"

β-function:

$$\boxed{\beta := \Lambda \partial_\Lambda g = -g^3/4\pi^2}$$

$$\rightsquigarrow \frac{1}{g^2} = \frac{1}{2\pi^2} \ln \varphi/\Lambda \qquad \varphi \quad \Lambda$$

Superconformal Ward-id:

$$\boxed{\mathcal{F}(A) - \frac{1}{2}\mathcal{F}'(A)\, A = \frac{i}{\pi} u}$$

Instanton-effects:

Vol moduli space

$$\mathcal{F}(A) = \mathcal{F}_{1\text{-Loop}} + \sum_{n=1}^{\infty} \mathcal{F}_n \left(\frac{\Lambda}{A}\right)^{4n} A^2$$

Symmetries

1) super conformal Ward ID

super conformal anomaly

scaling $\Rightarrow$ $$\boxed{\mathcal{F}(A) - \frac{1}{2}\mathcal{F}'(A)\,A = \frac{i}{\pi}u}$$

perturbatively: $u = 1/2\, A^2$ $\xleftarrow{\text{"2 dim"}}$ $A^2 \ln A^2/\Lambda^2$

2.) Symmetries of $\tau = \theta/2\pi + 4\pi i/g^2$

a) from index theorem: (p.89)

$$\frac{1}{32\pi^2}\int d^4x\, F_{\mu\nu}\tilde{F}_{\mu\nu} \in \mathbb{Z}$$

$\Rightarrow$ $\Theta \to 2\pi n + \Theta$ is symmetry

$$\boxed{\tau \to \tau + 1}$$

b) Duality ($U(1)$)

$$\mathcal{L}_{U(1)} = \frac{1}{32\pi} \operatorname{Im} \int \tau(a) \left(F + i\tilde{F} \right)^2 d^4x$$

dual description: F field

Bianchi-id with Lagrange multiplier A_D

$$\frac{1}{8\pi} \int A_{D\mu} \underbrace{\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma}}_{\text{Bianchi } = 0} d^4x$$

$$= \frac{1}{8\pi} \int \varepsilon^{\rho\sigma\mu\nu} \partial_\mu A_{D\nu} F_{\rho\sigma} d^4x$$

$$= \frac{1}{8\pi} \int \underbrace{\left(\frac{1}{2} \varepsilon^{\rho\sigma\mu\nu} \left(\partial_\mu A_{D\nu} - \partial_\nu A_{D\mu} \right) \right)}_{\tilde{F}_D{}^{\rho\sigma}} F_{\rho\sigma} d^4x$$

$$= \frac{1}{16\pi} \operatorname{Re} \int \left(\tilde{F}_D - iF_D \right) \left(F + i\tilde{F} \right) d^4x$$

$$\Rightarrow \mathcal{L}_{U(1)} = \frac{1}{32\pi} \operatorname{Im} \int \left(-\frac{1}{\tau} \right) \left(F_D + i\tilde{F}_D \right)^2 d^4x$$

$\Rightarrow$ a

$$\frac{1}{32\pi} \operatorname{Im} \int \tau(a) \left(F + i\tilde{F}\right)^2 d^4x$$

$$+ \frac{1}{16\pi} \operatorname{Re} \int \left(\tilde{F}_D - iF_D\right)\left(F + i\tilde{F}\right) d^4x$$

$$= \frac{1}{32\pi} \operatorname{Im} \int \tau(a) \left[\left(F + i\tilde{F}\right) + \frac{1}{\tau}\left(F_D + i\tilde{F}_D\right)\right]^2 d^4x$$

$$- \frac{1}{32} \operatorname{Im} \int \left(-\frac{1}{\tau}\right) \left(F_D + i\tilde{F}_D\right)^2 d^4x$$

$$= \frac{1}{16\pi^2} \operatorname{Im} \int \left(-\frac{1}{\tau}\right) \left(F_D + i\tilde{F}_D\right)^2 d^4x$$

$$\int dA \longrightarrow \int dF \qquad F = dA$$

elektr. $\Longleftrightarrow$ magnet.

Susy $\Rightarrow$ 122a

$$\Rightarrow \boxed{\tau \to -\frac{1}{\tau}}$$

Symmetry group:

generated by

$$\left.\begin{array}{l} \tau \to \tau + 1 \\ \tau \to -\frac{1}{\tau} \end{array}\right\} SL(2,\mathbb{Z})$$

$$\boxed{\tau \to \frac{a\tau + b}{c\tau + d} \quad \text{with} \quad \begin{array}{l} a,b,c,d \in \mathbb{Z} \\ ad - bc = 1 \end{array}}$$

p. 35: $\quad A_D = \frac{\partial \mathcal{F}}{\partial A}$.

$$\boxed{\begin{pmatrix} A_D \\ A \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_D \\ A \end{pmatrix}}$$

$N=1$ Susy:

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu$$

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D - \tfrac{i}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha F_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar\lambda)_\alpha$$

$$\partial_\mu \tilde F^{\mu\nu} = 0 \quad \to \quad \mathrm{Im}\, \mathcal{D}W = 0$$

<u>Lagrange multiplicator:</u>

$$\frac{1}{8\pi}\,\mathrm{Im}\int d^2\theta\, \tau(A)\, W^2$$

$$+ \frac{1}{4\pi}\,\mathrm{Im}\int d^4x\, d^2\theta\, d^2\bar\theta\, V_D\,\mathcal{D}W$$

$$\|$$

$$\left[\ \frac{1}{4\pi}\,\mathrm{Re}\int d^4x\, d^2\theta\, \underbrace{\underbrace{d^2\bar\theta\,(i\mathcal{D}V_D)}_{-\frac{1}{8}\bar D^2\mathcal{D}V_D}}_{W_D}\, W \qquad (-17)\right.$$

$$\left. = -\frac{1}{4\pi}\,\mathrm{Im}\int d^4x\, d^2\theta\, W_D W\ \right]$$

$$\to \frac{1}{8\pi}\,\mathrm{Im}\int d^2\theta \left(-\frac{1}{\tau(A)}\, W_D^2\right)$$

$$\Rightarrow \quad \left.\begin{array}{l} \tau \to \tau_D = -\dfrac{1}{\tau} \\ \text{und } \tau \to \tau + 1 \end{array}\right\} SL(2,\mathbb{Z})$$

(-30)

$$(\theta \to \theta + 2\pi)$$

$$\tau \to \frac{a\tau+b}{c\tau+d}$$

$a,b,c,d \in \mathbb{Z}$: $ad-bc=1$

Kinetic Term: $\frac{1}{4\pi}\,\mathrm{Im}\int d^2\theta\, d^2\bar\theta\, \frac{\partial \mathcal{F}}{\partial A}\,\bar A$

$$h(A) = \frac{\partial \mathcal{F}}{\partial A} \quad \text{with} \quad \frac{\partial h}{\partial A} = \tau$$

$$\tau_D = -1/\tau \quad \text{with} \quad \frac{\partial h_D}{\partial A_D} = -1/\tau$$

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and $\frac{1}{4\pi}\,\mathrm{Im}\int d^2\theta\, d^2\bar\theta\; h_D\, \bar A_D$

$$\Rightarrow \quad h_D = -A\,, \quad A_D = h \quad \left(= \frac{\partial \mathcal{F}}{\partial A}\right)$$

$$\Longrightarrow \quad \begin{pmatrix} A_D \\ A \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_D \\ A \end{pmatrix}$$

Solution (sketch) (A $N=2$ multiplet)

$$\mathcal{F}(A) - \frac{1}{2}\underbrace{\mathcal{F}'(A)}_{A_D}\, A = \frac{i}{\pi}\, u \qquad (*)$$

Uniformisation with u:

$\boxed{\begin{array}{l} (u \to \infty \quad u \simeq \tfrac{1}{2}A^2) \\ \circlearrowleft \ \ln u \to \ln u + 2\pi i \\ \quad \ln a \to \ln a + \pi i \\ \quad a_D \to -a_D + 2a \\ \quad a \to -a \end{array}}$

Derivative of $(*)$ w.r.t. u:

$$\underbrace{a_D \frac{\partial a}{\partial u} - \frac{\partial a_D}{\partial u}\, a}_{\text{Wronskian } W(a_D,\, a)} = \frac{2i}{\pi}$$

$\boxed{T^{-2} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}}$

$$a_D \frac{\partial a}{\partial u} - \frac{\partial a_D}{\partial u} a = W(a_D, a) = \text{konst}$$

$$\Rightarrow \quad a_D \frac{\partial^2 a}{\partial u^2} - \frac{\partial^2 a_D}{\partial u^2} a = 0$$

solved by: $a''_{(D)} + V(u) a_{(D)} = 0 \quad \left(\Rightarrow -a_D V a + V a_D a = 0\right)$

Theory of modular forms (conf. maps)

$$\left(\frac{a_D'}{a'} \Rightarrow\right) \tau(u) \xleftarrow{SL(2,\mathbb{Z})} = y_1/y_2 \qquad y'' + Q y = 0$$

mit $\quad Q = \frac{1}{2} \sum_i \left(\frac{1}{2} \frac{1-\alpha_i^2}{(u-u_i)^2} + \frac{\beta_i}{u-u_i} \right)$

und $0 \leq \alpha_i \leq 1$

$\Rightarrow$

$$(a_D', a') = v^{1/2} (y_1, y_2)$$

und $\quad v''/v - \frac{3}{2}\left(\frac{v'}{v}\right)^2 + 2V = 2Q \qquad (1)$

mit

$|u| \to \infty$:

$$Q \to \frac{1}{4u^2}$$

$$V \to 1/4u^2$$

V singular (bei u_s) $\Rightarrow$ Q singular (at u_i)

$\simeq V_s (u-u_s)^\gamma$

$$u_s = u_{i_c}$$

with (1) we get

$$-\frac{1}{2}\gamma(\gamma+2)\frac{1}{(u-u_s)^2} + 2V_s(u-u_s)^\gamma$$

$$\Rightarrow \quad V_s = \frac{1}{4}(1-\alpha_i^2) \qquad \gamma = -2$$

$$\text{or} \quad \gamma = -1 \pm \alpha_i \qquad \gamma > -2$$

$$\Rightarrow V(u) = v \prod_r (u-u_r)^{-2} \prod_s (u-u_s)^{-1\pm\alpha_s}$$

2 pole (dyons, monopole)

$\Rightarrow$ 2 simple poles

$$\Rightarrow V(u) = \frac{1}{4(u-a)(u-b)} \qquad a=-b=1$$

$$\Rightarrow Q = \frac{1}{2}\left(\frac{1}{(u-1)^2} + \frac{1}{(u+1)^2}\right) - \frac{1/2}{(u-1)(u+1)}$$

$$\boxed{\begin{aligned} \Rightarrow a(u) &= \sqrt{2}\,\sqrt{u+1}\; F\left(-\tfrac{1}{2}, \tfrac{1}{2}, 1; \frac{2}{u+1}\right) \\ a_D(u) &= \frac{i(u-1)}{2}\, F\left(\tfrac{1}{2}, \tfrac{1}{2}, 2; \frac{1-u}{2}\right) \end{aligned}}$$

$$\boxed{\tau = a_D'/a'}$$

$$T^{2}\,\Gamma = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \qquad u = 1$$

$$T^{-2} = \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \qquad u \to \infty$$

<u>Monodromy group</u> : generated by

$$\left[\quad M = \{ T^2\Gamma, T^{-2} \} \quad \right]$$

$$u \to 1 - \frac{4}{1-u}$$

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} -a \\ a_D \end{pmatrix}$$

<u>N=1 Supergrav.</u> :

$$\hat{W} = \sqrt{2}\, A_D\, M \tilde{M} + m\, U(A_D)$$

$a_D M = a_D \tilde{M} = 0$, $M\tilde{M} + m \frac{\partial U}{\partial A_D} = 0$, $a_D = 0$, $\frac{\partial U}{\partial a_D} \neq 0 \Rightarrow M \neq 0$ Nos

Dualität ($U(1)$)

1.) Reine Eichfeldwirkung

$$\mathcal{L}_{U(1)} = \frac{1}{32\pi}\,\mathrm{Im}\int \tau(a)\,(F + i\tilde{F})^2\,d^4x$$

$$= \frac{1}{16\pi^2}\,\mathrm{Im}\int \tau(a)\left(F^2 + i\tilde{F}F\right)$$

$\uparrow$

$\tilde{F}^2 = -F^2\ ,\ \tilde{\tilde{F}} = F$

Duale Beschreibung: F Feld

Bianchi identität mit Lagrangemultiplikator

$$\frac{1}{8\pi}\int A_{D\mu}\,\underbrace{\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}}_{\substack{\text{Bianchi identität } = 0\\ \text{Monopol } \varepsilon^{0\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 8\pi\,\delta^{(4)}(x))}}\,d^4x$$

$$= \frac{1}{8\pi}\int \varepsilon^{\rho\sigma\mu\nu}\partial_\mu A_{D\nu}\,F_{\rho\sigma}\,d^4x$$

$$= \frac{1}{8\pi}\int \underbrace{\frac{1}{2}\varepsilon_{\rho\sigma\mu\nu}\left(\partial_\mu A_{D\nu} - \partial_\nu A_{D\mu}\right)}_{\tilde{F}_{D\,\rho\sigma}}\,F^{\rho\sigma}\,d^4x$$

$$= \frac{1}{16\pi}\,\mathrm{Re}\int (\tilde{F}_D - iF_D)(F + i\tilde{F})\,d^4x$$

Gaußsche Integration über F (klassische Bewegungsgl.)

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$$\mathcal{L}_{U(1)} \rightarrow \frac{1}{32\pi}\,\mathrm{Im}\int\left(-\frac{1}{\tau}\right)(F_D + i\tilde{F}_D)^2\,d^4x$$

$$= \frac{1}{16\pi}\int\left(-\frac{1}{\tau}\right)\left(F_D^2 + i\tilde{F}_D F_D\right)$$

Instanton effekte

$$\int dA_\mu \, e^{-S[A, \dots]}$$

$$= \sum_n \int da_\mu^{(n)} \, e^{-S[A_n(s) + a_\mu^{(n)}, \dots]} \; d\mu(s)$$

Moduli

$$F_{\mu\nu}(A_n) = \pm \tilde{F}_{\mu\nu}(A_n)$$

und $\frac{1}{32\pi^2}\int F\tilde{F} = n$

— Tatsächlich gibt es $\underbrace{8|n|-3}_{\text{Moduli parameter } s}$ Instantone $(SU(2))$

$8n - 3 = n(4$ Translationen $+ 1$ Dilatation

$+ \underset{SU(2)}{3}$ Eichrotationen$) - 3$ globale Eichrotationen

$$e^{-S[A_n]} = e^{-8\pi^2|n|/g^2} \longleftarrow \frac{1}{g^2} = \frac{1}{2\pi^2}\ln a/\Lambda$$

Seiberg '88: keine Inst.-Antiinst. effekte! (holomorphie)

$$= \left(\frac{\Lambda}{a}\right)^{4|n|}$$

$$\Rightarrow \boxed{\mathcal{F} = \frac{i}{2\pi} A^2 \ln A^2/\Lambda^2 + \sum_{n=1}^{\infty} \mathcal{F}_n \left(\frac{\Lambda}{A}\right)^{4n} A^2}$$