Exercise 1

1) *Path integral for the harmonic oscillator*

a) The action for the harmonic oscillator is given by

$$S = \int_{t_a}^{t_b} \left(\frac{m}{2}\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 \right) dt \ . \tag{1}$$

Calculate the transition amplitude $\langle x_b | e^{-iHt} | x_a \rangle = \int \mathcal{D}x \, e^{iS[x]}$ using the path integral method.

Hint: Any trajectory $x(t)$ can be written as sum of the classical path $x_{cl}(t)$ and a fluctuation $\delta x(t)$, i.e. $x(t) = x_{cl}(t) + \delta x(t)$, where $\delta x(t_a) = \delta x(t_b) = 0$. Show that the amplitude can be written as $\langle x_b | e^{-iHt} | x_a \rangle = F(t) \times e^{iS[x_{cl}]}$ where the prefactor $F(t)$ contains the contribution from the fluctuations around the classical solution, and evaluate the phase factor $e^{iS[x_{cl}]}$.

b) Continue the solution to imaginary time $\tau = it$. Evaluate the trace assuming periodic boundary conditions, and let $\omega\tau \gg 1$. What form does the transition amplitude take?

2) *Anharmonic oscillator and tunneling solution*

a) Consider the anharmonic oscillator with the Euclidean action

$$S_E = \int \left(\frac{m}{2}\dot{x}^2 + \frac{1}{2}m\omega^2 x^2 + \lambda x^4 \right) d\tau \ . \tag{2}$$

Expand the path integral in powers of λ and calculate the ground state energy to two loop order.

b) Consider now the anharmonic oscillator with a double-well potential,

$$V(x) = \lambda \left(x^2 - \eta^2 \right)^2 \ . \tag{3}$$

Solve the equation of motion in imaginary time and show that this yields the classical tunneling (instanton) solution $x_{cl}(\tau)$ with $\omega^2 = 8\lambda\eta^2$,

$$x_{cl}(\tau) = \eta \tanh\left[\frac{\omega}{2}(\tau - \tau_0) \right] \ . \tag{4}$$

c)* Determine the tunneling amplitude $\langle -\eta | e^{-H\tau} | \eta \rangle$ in analogy to exercise 1a) and consider the contribution of the fluctuations about the instanton solution.