Exercise 2

1) *Solitons in the sine-Gordon model*

The sine-Gordon equation is given by

$$\Box\phi + \mu^2/\lambda \sin\lambda\phi = 0 \ , \tag{1}$$

where μ is the mass parameter, and λ is the coupling. It is convenient to work in light-cone coordinates (u, v), where $u = \frac{1}{2}(x^1 + x^0)$, and $v = \frac{1}{2}(x^1 - x^0)$, and the d'Alembertian is given by $\Box \equiv -\partial_u\partial_v$.

a) Show that it is possible to establish a relationship between two independent solutions ϕ_1 and ϕ_2 of eq. (1) in the form of two coupled ordinary differential equations (ODEs):

$$\partial_u(\phi_1 + \phi_2) = f(\phi_1 - \phi_2) \ , \tag{2}$$

$$\partial_v(\phi_1 - \phi_2) = g(\phi_1 + \phi_2) \ . \tag{3}$$

Determine the functions f and g!

b) Integrate the above system of ODEs to determine the 1-soliton solution. Show that it takes the form

$$\phi = 4/\lambda \ \arctan\exp\left(\mu\alpha\, u + \mu/\alpha\, v + \delta\right) \ . \tag{4}$$

What is the significance of the parameters α and δ?

2) *Two kinks meet ...*

a) Find the 2-soliton (kink-kink) solution in analogy to exercise **1b)**.

b) What are the asymptotic states at large times for the 2-soliton solution? What happens in the scattering region?