Exercise 3

1) *Fermions in a soliton background*

Consider a theory of a scalar field ϕ and spinor field ψ in $1+1$ dimensions given by the Lagrangian density

$$\mathcal{L}(\phi, \psi) = \mathcal{L}_\phi + \bar{\psi}\,(i\partial\!\!\!/ - g\phi)\,\psi \ , \tag{1}$$

where $\mathcal{L}_\phi$ depends only on ϕ. Suppose that in the absence of fermions the classical field equation for ϕ possesses a static, finite-energy solution ϕ_c which satisifes $\phi_c(-\infty) = -\phi_c(\infty)$.

a) Show that the spectrum of the Hamiltonian corresponding to (1) has a zero mode which is self-conjugate under charge conjugation.

b) Assume the existence of a one-soliton state $|p, m; \pm\rangle$ characterized by momentum p, and mass m, where $\pm$ denotes the degeneracy required by the zero-energy fermion solution. Calculate the fermion number in the one-soliton state!

2) *Complex scalar in $d+1$ dimensions*

a) Now, consider a complex scalar field ϕ in one spatial dimension with Lagrangian density

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - \frac{\lambda}{2}\left(v^2 - |\phi|^2\right)^2 \ . \tag{2}$$

The classical equation of motion

$$\Box\phi + \lambda\left(v^2 - |\phi|^2\right)\phi = 0 \ , \tag{3}$$

has a (real) static, finite-energy solution. Show that it is unstable with respect to small perturbations.

b) In $2+1$ dimensions the complex scalar field (2) has a static vortex solution with logarithmically divergent energy. Replace the partial derivative by the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu + ie\mathcal{A}_\mu \ , \tag{4}$$

and show that it is possible to cancel the logarithmic divergence by a gauge potential $\mathcal{A}_\mu$ that is asymptotically pure gauge. Calculate the magnetic flux of the vortex solution!