Exercise 5

1) *Fundamental group*

a) Recall the definition of the product between two loops α_x and β_x at $x \in \mathcal{M}$ denoted by $\alpha_x * \beta_x$ (see lecture). Verify that the product of homotopy classes $[\alpha_x]$ and $[\beta_x]$, as given by

$$[\alpha_x] * [\beta_x] = [\alpha_x * \beta_x] \ , \tag{1}$$

is well defined and satisfies the group axioms.

b) Let $\mathcal{M}$ be arcwise connected. Show that the fundamental group $\pi_1(\mathcal{M}, x)$ is independent of the base point $x \in \mathcal{M}$, i.e.

$$\pi_1(\mathcal{M}, x) \cong \pi_1(\mathcal{M}, y) \ , \quad \forall x, y \in \mathcal{M} \ . \tag{2}$$

1) *Higher homotopy groups and Hopf invariant*

In 1931 Hopf showed that the third homotopy group of the two-sphere $\pi_3(S^2)$ is non-trivial. As an example he considered what is now known as the *Hopf map* $\pi : S^3 \to S^2$, defined by

$$y^1 = 2\left(x^1 x^3 + x^2 x^4\right) \ , \tag{3}$$

$$y^2 = 2\left(x^2 x^3 - x^1 x^4\right) \ , \tag{4}$$

$$y^3 = (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2 \ , \tag{5}$$

where the 3-sphere S^3 is embedded in $\mathbb{R}^4$, i.e.

$$S^3 = \left\{x \in \mathbb{R}^4 \,|\, (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1\right\} \ , \tag{6}$$

and the 2-sphere S^2 is parametrized by

$$(y^1)^2 + (y^2)^2 + (y^3)^2 = 1 \ . \tag{7}$$

Let $\{U_N, U_S\}$ be an open covering of S^2, with U_N (U_S) being the northern (southern) hemisphere. Their intersection $U_N \cap U_S = S^1$ is simply the equator (compare Fig.1). For both U_N and U_S in the open cover we have a *local trivialization* $\phi_i : U_i \times U(1) \to \pi^{-1}(S^2)$, $i = N, S$. ϕ_N and ϕ_S are related on the intersection $U_N \cap U_S$, where

$$\phi_N(p, g) = \phi_S\left(p, t_{NS}(p) \circ g\right) \ , \quad p \in U_N \cap U_S = S^1 \ , \ g \in U(1) \ , \tag{8}$$

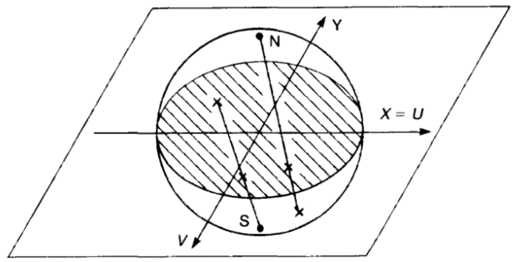

Figure 1: Stereographic projection of the sphere S^2.

and the *transition function* is given by $t_{NS} : S^1 \to U(1)$.

a) Find the local trivializations $\phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times U(1)$, $i = N, S$.

Hint: Use the coordinates (X, Y) (and (U, V)) of the stereographic projection with respect to the north (south) pole to express $Z = X + iY$ (and $W = U + iV$) in terms of the complex coordinates $z^1 = x^1 + ix^2$ and $z^2 = x^3 + ix^4$ of the 3-sphere S^3.

b) Show that the map $\pi : S^3 \to S^2$ belongs to the element 1 of $\pi_3(S^2) \cong \pi_1(U(1)) = \mathbb{Z}$.