QCD

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I. BASICS OF QCD

The theory of strong interactions, quantum chromodynamics (QCD), has been put-forward on the basis of scattering experiments that showed an internal $SU(3)$-symmetry and related charges much the same way quantum-electrodynamics (QED) shows the $U(1)$-symmetry related to the electric charge. The corresponding gauge theory, $SU(3)$ Yang-Mills theory, is non-Abelian and hence self-interacting, i.e. the (quantised) pure gauge theory is already non-trivial in contradistinction to the pure $U(1)$-theory.

A. Yang-Mills theory

1. Classical action

As in QED, the classical action can be derived from the gauge-invariant (minimal) extension of the action of a free spin-one particle. The requirement of invariance of physics under local $SU(N_c)$-rotations with $\mathcal{U} \in SU(N_c)$ (and the minimal coupling) leads us from partial to covariant derivatives,

$$\partial_\mu \to D_\mu(A) = \partial_\mu - i\, g\, A_\mu\,. \tag{1}$$

The gauge field A_μ is Lie-algebra–valued,

$$A_\mu = A_\mu^a\, t^a\,, \qquad \text{with} \qquad a = 1, ..., N_c^2 - 1\,. \tag{2}$$

where t^a are the generators of $SU(N)$ with

$$[t^a, t^b] = i\, f^{abc} t^c\,, \qquad\qquad \mathrm{tr}_\mathrm{f}\,(t^a t^b) = \frac{1}{2}\delta^{ab}\,, \tag{3}$$

where tr_f is the trace in the fundamental representation and the expansion coefficients f^{abc} are the structure constants of the Lie algebra. In the adjoint representation the covariant derivative (1) reads

$$D_\mu^{ab}(A) = \partial_\mu \delta^{ab} - g\, f^{abc} A_\mu^c\,, \qquad \text{with} \qquad (t^c_{\mathrm{ad}})^{ab} = -i\, f^{abc}. \tag{4}$$

The covariant derivative D_μ has to transform as a tensor under gauge transformations,

$$D_\mu(A) \to D_\mu(A^{\mathcal{U}}) = \mathcal{U}\, D_\mu\, \mathcal{U}^\dagger\,, \qquad \text{with} \qquad \mathcal{U} = e^{i\omega} \in SU(N_c)\,, \tag{5}$$

where $\omega \in su(N_c)$ is the corresponding Lie algebra element. This implies

$$A_\mu \to A_\mu^{\mathcal{U}} = \mathcal{U}\, A_\mu\, \mathcal{U}^\dagger - \frac{i}{g}\mathcal{U}\, \partial_\mu \mathcal{U}^\dagger\,, \tag{6}$$

Consequently, in a non-Abelian gauge theory the gauge boson A_μ carries the corresponding color charge. In physical QCD we have $N_c = 3$, the gauge group has eight generators, $a = 1, ..., 8$, the Gell-Mann matrices. Often one also considers the $SU(2)$-theory as it has the same qualitative features (asymptotic freedom and confinement) but is technically simpler. There are various notations on the market leading to factors i and $-$ in the Lie algebra relations above. In the present lecture notes we have chosen hermitian generators which leads to the factor $+1/2$ for the trace in (3). It also entails real structure constants f^{abc} in the Lie-algebra in (3).

The fieldstrength tensor is defined analoguously to QED with the commutator of covariant derivatives. Due to (1) and (3) we find

$$F_{\mu\nu} = \frac{i}{g}[D_\mu\,,\, D_\nu]\,, \tag{7}$$

and the fieldstrength tensor $F_{\mu\nu}$ can be computed as

$$F_{\mu\nu} = F_{\mu\nu}^a t^a\,, \qquad \text{with} \qquad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\, f^{abc} A_\mu^b\, A_\nu^c. \tag{8}$$

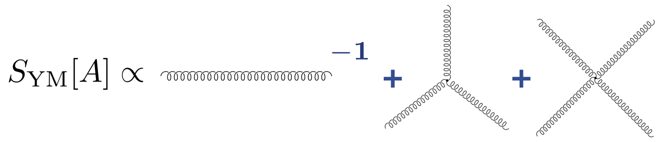


FIG. 1: Diagrammatical form of the Yang-Mills action.

It is easily shown with (7) that the fieldstrength $F_{\mu\nu}$ transforms covariantly (as a tensor) under gauge transformations,

$$\begin{aligned} F_{\mu\nu}(A^{\mathcal{U}}) &= \frac{i}{g}\,[\mathcal{D}_\mu(A^{\mathcal{U}}_\mu), \mathcal{D}_\nu(A^{\mathcal{U}}_\nu)] \\ &= \mathcal{U}\,\frac{1}{g}[\mathcal{D}_\mu(A_\mu), \mathcal{D}_\nu(A_\nu)]\,\mathcal{U}^\dagger \\ &= \mathcal{U}\,F_{\mu\nu}(A)\,\mathcal{U}^\dagger. \end{aligned} \tag{9}$$

This leads us finally to the Yang-Mills (YM) action,

$$S_{\mathrm{YM}}[A] = \frac{1}{2}\int_x \mathrm{tr_f}\, F_{\mu\nu}\, F_{\mu\nu} = \frac{1}{4}\int_x F^a_{\mu\nu}\, F^a_{\mu\nu}\,, \tag{10}$$

with $\int_x = \int d^dx$. Its gauge invariance follows from (9),

$$S_{\mathrm{YM}}[A^{\mathcal{U}}] = \frac{1}{2}\int_x \mathrm{tr_f}\, \mathcal{U}\, F_{\mu\nu}(A) F_{\mu\nu}(A)\, \mathcal{U}^\dagger = S_{\mathrm{YM}}[A]\,, \tag{11}$$

where the last equality holds due to cyclicity of the trace in color space. Clearly the action (10) with (8) is a self-interacting theory with coupling constant g. It has a quadratic kinetic term and three-gluon and four-gluon vertices. This is illustrated diagrammatically by Fig. 1. This allows us to read-off the Feynman rules for the purely gluonic vertices. The full Feynman rules of QCD in the general covariant gauge are summerised in Fig. 23. As in QED we identify color-electric and color-magnetic fields as the components in the fieldstrength tensor,

$$\begin{aligned} E^a_i &= F^a_{0i} \\ B^a_i &= \frac{1}{2}\epsilon_{ijk} F^a_{jk}. \end{aligned} \tag{12}$$

In contradisctinction to QED the above color-electric and magentic fields are no observables, they change under gauge transformations. Only $\mathrm{tr}\,\vec{E}^2, \mathrm{tr}\,\vec{B}^2$ are observables.

2. Generating functional of Yang-Mills theory

The quantisation of Yang-Mills theory is done within the path integral. Naively, the generating functional of pure YM-theory would read

$$Z[J] = \int dA \exp\left(-S_{YM}[A] + \int_x J^a_\mu\, A^a_\mu\right), \tag{13}$$

which contains redundent integrations due to gauge invariance of the action, see (11). These redundant integrations are usually removed by introducing a gauge fixing,

$$\mathcal{F}[A_{\mathrm{gf}}] = 0 \tag{14}$$

Often used gauge fixings are provided by

$$\begin{aligned} \partial_\mu A_\mu &= 0\,, \qquad \text{covariant or Lorentz gauge}\,,\\ \partial_i A_i &= 0\,, \qquad \text{Coulomb gauge}\,,\\ n_\mu A_\mu &= 0\,, \qquad \text{axial gauge}\,. \end{aligned} \tag{15}$$

The general covariant gauge has the technical advantage that it does not single out a space-time direction. This property reduces the possible tensor structure of correlation functions and hence simplifies computations. The Coulomb gauge and the axial gauge single out specific frames. At finite temperature (and density) this might be useful as the temperature singles out the thermal rest frame. It is here were the Coulomb gauge and the temporal or Weyl gauge ($n_\mu = \delta_{\mu 0}$) is used often.

Gauge fields that are connected by gauge transformations are physically equivalent, i.e. their actions agree. They lie in so-called gauge orbits, $\{A^{\mathcal{U}}, \mathcal{U} \in SU(N)\}$, and fixing a gauge is equivalent to choosing a representative of such an orbit $A \to A_{\rm gf}$, up to potential (Gribov) copies. The occurance of Gribov copies and how to handle them is discussed in Appendix B. To keep things simple we ignore them for the time being and proceed with

The path integral measure dA in (13) can be split into an integration over physically inequivalent configurations $A_{\rm gf}$ and the gauge group,

$$\mathrm{d}A = \mathrm{d}A_{\rm gf}\, \mathrm{d}\mathcal{U} \cdot J[A_{\rm gf}], \tag{16}$$

where $J[A_{\rm gf}]$ is the Jacobian of the transformation $A \to (A_{\rm gf}, \mathcal{U})$ and d$\mathcal{U}$ is the Haar measure of the gauge group, see e.g. [**?**]. The coordinate transformation (16) and the computation of the Jacobian J is done within the Faddeev-Popov quantisation, [1]. Within this procedure one inserts unity in the path integral,

$$1 = \int \mathrm{d}\mathcal{U}\, \delta\left[\mathcal{F}[A^{\mathcal{U}}]\right]\, \Delta_{\mathcal{F}}[A] \quad \text{with} \quad \Delta_{\mathcal{F}}[A] = \left(\int \mathrm{d}\mathcal{U}\, \delta\left[\mathcal{F}[A^{\mathcal{U}}]\right]\right)^{-1}, \tag{17}$$

where $\Delta_{\mathcal{F}}[A]$ is gauge-invariant due to the property $\mathrm{d}(\mathcal{U}\mathcal{V}) = \mathrm{d}\mathcal{U}$ of the Haar measure. Let us consider a general observable $\mathcal{O}$, like *e.g.* $\mathrm{tr}F^2(x)\,\mathrm{tr}F^2(0)$. The expectation value of this observable can be calculated by

$$\langle \mathcal{O} \rangle = \frac{\int \mathrm{d}A\, \mathcal{O}[A]\, e^{-S_{\rm YM}[A]}}{\int \mathrm{d}A\ e^{-S_{\rm YM}[A]}} = \frac{\int \mathrm{d}A\, \mathrm{d}\mathcal{U}\, \delta\left[\mathcal{F}[A^{\mathcal{U}}]\right]\, \Delta_{\mathcal{F}}[A]\, \mathcal{O}[A]\, e^{-S_{\rm YM}[A]}}{\int \mathrm{d}A\, \mathrm{d}\mathcal{U}\, \delta\left[\mathcal{F}[A^{\mathcal{U}}]\right]\, \Delta_{\mathcal{F}}[A]\, e^{-S_{\rm YM}[A]}}\,, \tag{18}$$

where we have simply inserted (17) into the path integral. In (18) all terms are gauge invariant except for the δ-function. Hence we can absorb the $\mathcal{U}$-dependence via $A \to A^{\mathcal{U}^\dagger}$. Then the (infinite) integral over the Haar measure decouples in numerator and denominator, and we arrive at

$$\langle \mathcal{O} \rangle \; = \; \frac{\int \mathrm{d}A\, \delta\left[\mathcal{F}[A]\right]\, \Delta_{\mathcal{F}}[A]\, \mathcal{O}[A]\, e^{-S_{\rm YM}[A_{\rm gf}]}}{\int \mathrm{d}A\, \delta\left[\mathcal{F}[A]\right]\, \Delta_{\mathcal{F}}[A]\, e^{-S_{\rm YM}[A]}}\,,$$

It is left to compute the Jacobian $J[A] = \Delta_{\mathcal{F}}[A]$. To that end we use the representation of the Dirac δ-function

$$\delta[\mathcal{F}[A^{\mathcal{U}}]] = \frac{1}{|\det \frac{\delta \mathcal{F}}{\delta \omega}|}\delta[\omega - \omega_1] \quad \text{with} \quad \mathcal{U} = e^{i\omega} \quad \text{and} \quad \mathcal{F}[A_{\rm gf} = A^{\mathcal{U}(\omega_1)}] = 0\,. \tag{19}$$

This leads to

$$\Delta_{\mathcal{F}}[A] = |\det \mathcal{M}_{\mathcal{F}}[A_{\rm gf}]| \qquad \text{with} \qquad \mathcal{M}_{\mathcal{F}}[A] = \left.\frac{\delta \mathcal{F}}{\delta \omega}\right|_{\omega=0} [A]\,. \tag{20}$$

where $A_{\rm gf}$ is the solution with the minimal distance to $A = 0$. In (20) we introduced the Faddeev-Popov determinant $\det\mathcal{M}_{\mathcal{F}}$. For the Landau gauge,

$$\partial_\mu A^a_\mu = 0\,, \qquad a = 1, ..., N_c^2 - 1\,, \tag{21}$$

we get

$$g\,\mathcal{M}_{\mathcal{F}}[A] = -\frac{\delta \partial_\mu D_\mu \omega}{\delta \omega} = -\partial_\mu D_\mu \mathbb{1}. \tag{22}$$

where we have used that the gauge transformation of the gauge field, (6) reads for infinitesimal gauge transformations $\mathcal{U} = 1 + i\,\omega$,

$$A_\mu^{\mathcal{U}} = A_\mu - \frac{1}{g} D_\mu \omega\,. \tag{23}$$

Note that the factor $1/g$ in the definition cancels in the normalised expectation values and we drop it. Furthermore we assume that $-\partial_\mu D_\mu$ is a positive definite operator and we arrive at

$$\Delta_{\mathcal{F}}[A] = \det\left(-\partial_\mu D_\mu\right)\,, \quad \text{or generally} \quad \Delta_{\mathcal{F}}[A] = \det \mathcal{M}[A]\,. \tag{24}$$

For the final expression for the generating functional (13) we slightly modify the gauge by introducing a Gaußian average over the gauges

$$\delta[\mathcal{F}[A^{\mathcal{U}}]] \to \int \mathrm{d}\mathcal{C}\, \delta[\mathcal{F}[A^{\mathcal{U}} - \mathcal{C}]] \exp\left\{-\frac{1}{2\xi}\int_x \mathcal{C}^a \mathcal{C}^a\right\}\,. \tag{25}$$

This leads us to the gauge fixed generating functional

$$Z[J] = \int \mathrm{d}A\, \Delta_{\mathcal{F}}[A]\, e^{-S_{\mathrm{YM}}[A] + \frac{1}{2\xi}\int_x \mathcal{F}^a \mathcal{F}^a}\,, \tag{26}$$

The Faddeev-Popov determinant, (24), can be represented by means of a Grassmann integration,

$$\det \mathcal{M}_{\mathcal{F}}[A] = \int \mathrm{d}C\, \mathrm{d}\bar{C}\, \exp\left\{\int\!\!\int d^dx\, d^dy\, \bar{C}^a(x) \mathcal{M}_{\mathcal{F}}^{ab}(x,y) C^b(y)\right\}\,. \tag{27}$$

Restricting ourselves to the averaged Landau gauge, (21) with (25) we finally arrive at

$$Z[J_A, J_C, \bar{J}_C] = \int \mathrm{d}A\, \mathrm{d}C\, \mathrm{d}\bar{C}\, e^{-S[A,c,\bar{c}] + \int_x \left(J_A\cdot A + \bar{J}_C\cdot C - \bar{C}\cdot J_C\right)}\,, \tag{28}$$

with

$$S_A = \frac{1}{4}\int_x F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2\xi}\int_x \left(\partial_\mu A_\mu^a\right)^2 + \int_x \bar{C}^a \partial_\mu D_\mu^{ab} C^b\,, \tag{29}$$

where $\int_x = \int d^dx$ and the Landau gauge is achieved for $\xi = 0$. Note that the ghost action implies a negative dispersion for the ghost, related to the determinant of the positive operator $\mathcal{M}_{\mathcal{F}} = -\partial_\mu D_\mu$. However, this is a matter of convention, we might as well use a positive dispersion, the minus sign drops out for all correlation functions which do not involve ghosts, and only those are related to scattering amplitudes. The source term reads in full details

$$\int_x \left(J_A\cdot A + \bar{J}_C\cdot C - \bar{C}\cdot J_C\right) = \int_x \left(J_{A,\mu}^a A_\mu^a + \bar{J}_C^a C^a - \bar{C}^a J_C^a\right)\,. \tag{30}$$

The Feynman rules derived from (29) are summarised in Appendix A.

B. QCD

1. *Classical action of the matter sector*

The classical action of the matter sector of QCD is given by the Dirac action of the quarks,

$$S_{\mathrm{Dirac}}[\psi, \bar{\psi}, A] = \int_x \bar{\psi}\cdot\left(\not{D} + m_\psi + \mu\gamma_0\right)\cdot\psi\,, \tag{31}$$

where

$$\not{D} = \gamma_\mu D_\mu \,, \quad \text{with} \quad \{\gamma_\mu \,, \, \gamma_\nu\} = 2\,\delta_{\mu\nu} \,. \tag{32}$$

In (31), the fermions carry Dirac ξ, gauge group indices A (fundamental representation) as well flavour indices f, that is $\psi^A_{\xi,f}$. The Dirac operator $\not{D}^{AB}_{\xi\xi'}$ is diagonal in the flavour space as is the chemical potential term. The mass term depends on the current quark masses related to spontanous symmetry breaking of the Higgs sector of the Standard Model and the mass matrix m^{ef}_ψ with $e, f = 1, ..., N_f$ is slightly off-diagonal (CKM-matrix). The up and down current quark masses are of the order $2-5$ MeV whereas the current quark mass of the strange quark is of the order 10^2 MeV. The other quark masses are of order $1 - 2 * 10^2$ GeV. In low energy QCD this has to be compared with the scale of strong chiral symmetry breaking $\Delta m \approx 300$ MeV. This situation is summarised in Table I. Evidently for most

Generation	first	second	third	Charge
Mass [MeV]	1.5-4	1150-1350	170×10^3	
Quark	u	c	t	$\frac{2}{3}$
Quark	d	s	b	$-\frac{1}{3}$
Mass [MeV]	4-8	80-130	$(4.1\text{-}4.4)\times10^3$	

TABLE I: Quark masses and charges. The scale of strong chiral symmetry breaking is $\Delta m \approx 300$ MeV as is $\Lambda_{\rm QCD}$. This entails that only $2+1$ flavours have to be considered for most applications to the phase diagram of QCD.

applications to the phase diagram of QCD we only have to consider the three lightest quark flavours, that is up, down and strange quark, to be dynamical. The current quark masses of up and down quarks are two order of magnitude smaller than the QCD infrared scales $\Lambda_{\rm QCD}, \Delta m, T_{\rm conf}, T_\chi$. Note that all of the latter scales in fact related to $\Lambda_{\rm QCD}$. Hence the up and down quarks can be considered to be massless. This leads to the important observation that the physical masses of neutrons and protons –and hence the masses of the world around us– comes about from strong chiral symmetry breaking and has but nothing to do with the Higgs sector.

In turn, the mass of the strange quark is of the order of $\Lambda_{\rm QCD}$ and has to be considered as heavy for application in low energy QCD. The three heavier flavours, charm, bottom and top, are essentially static they do not contribute to the QCD dynamics relevant for its phase structure even though in particular the c-quark properties and bound states are much influenced by the infrared dynamics of QCD. In summary we will consider the $N_f = 2$ and $N_f = 2 + 1$ flavour cases for the phase structure of QCD, while for LHC physics all flavours are relevant.

2. *Generating functional of QCD and perturbation theory*

The generating functional of QCD is the straightforward extension of that of Yang-Mills as presented in the previous Section I A, see (28). The quark fields are Grassmannian due to their fermionic nature and we are led to the generating functional

$$Z[J] = \int d\phi \, e^{-S_{\rm QCD}[\phi] + \int_x J\cdot\phi} \,, \tag{33}$$

The gauge fixed action $S_{\rm QCD}$ in (33) in the Landau gauge is given by

$$S_{\rm QCD}[\phi] \;=\; \frac{1}{4}\int_x F^a_{\mu\nu}F^a_{\mu\nu} + \frac{1}{2\xi}\int_x (\partial_\mu A^a_\mu)^2 + \int_x \bar c^a \partial_\mu D^{ab}_\mu c^b + \int_x \bar\psi\cdot(\not{D} + m_\psi + i\mu\gamma_0)\cdot\psi \,,$$

and we have introduced super-fields and super-currents

$$\phi = (A, C, \bar C, \psi, \bar\psi) \,, \qquad J = (J_A, J_C, \bar J_C, J_\psi, \bar J_\psi) \,, \tag{34}$$

with

$$d\phi = \int dA \, dC \, d\bar C \, d\psi \, d\bar\psi \,, \quad \text{and} \quad J\cdot\phi = J_A\cdot A + \bar J_C\cdot C - \bar C\cdot J_C + \bar J_\psi\cdot\psi - \bar\psi\cdot J_\psi \,. \tag{35}$$

The action in (34) is illustrated diagrammatically by Fig. 2, the gauge dependence being displayed by the last two graphs in the first line, the ghost terms, as well as the hidden gauge fixing dependence of the inverse gluon propagator.

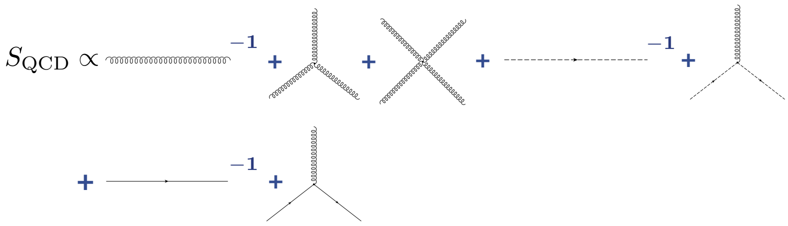


FIG. 2: Diagrammatical form of the QCD action.

The Feynman rules are summarised in Appendix A. The two equations (33), (34) define the fundamental quantum theory of strong interactions and have, apart from the mass matrix m_q of the quarks one input parameter, the strong coupling. In the full quantum theory we have a running coupling

$$\alpha_s(p) = \frac{g^2}{4\pi}\,, \tag{36}$$

where p is the relevant momentum/energy scale of a given process. The scale-dependence of $\alpha_s(p)$ is inflicted by quantum corrections. For perturbation theory being applicable the expansion parameter $\alpha_s/(4\pi)$ should be small. Moreover, the perturbative expansion is an asymptotic series (with convergence radius $\alpha_{s,\max} = 0$). The gluon self-coupling in QCD, depicted in Fig. 2 leads to a running coupling which decreases with the momentum scale, i.e. ,

$$\beta_g = \frac{1}{2} p\partial_p \alpha_s = -\beta_0 \alpha^2 + O(\alpha_s^3) \qquad \text{with} \qquad \beta_0 = \frac{\alpha_s^2}{12\pi}\left(11N_c + 2N_f\right)\,. \tag{37}$$

Integrating the β-function (37) at one loop leads to the running coupling

$$\alpha_s(p) = \frac{\alpha_s(\mu)}{1 + \beta_0 \alpha_s(\mu) \log \frac{p^2}{\mu^2}} + O(\alpha_s^2)\,, \tag{38}$$

with some reference (momentum) scale μ^2. The running coupling in (38) tends to zero logarithmically for $p \to \infty$. This property is called asymptotic freedom (Nobel prize 2004) and guarantees the existence of the the perturbative expansion of QCD. Its validity for large energies and momenta is by now impressively proven in various scattering experiments, see e.g. [2]. These experiments can also be used to define a running coupling (which is not unique beyond two loop, see e.g. [3]) and the related plot of $\alpha_s(p^2)$ in Fig. 3 has been taken from [2].

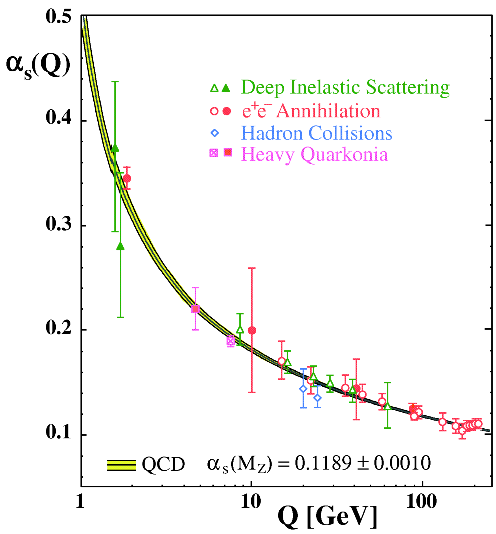
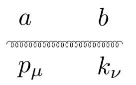


FIG. 3: Experimental tests of the running coupling, figure taken from [2].

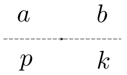
In turn, in the infrared regime of QCD at low momentum scales, perturbation theory is not applicable any more. The coupling grows and the failure of perturbation theory is finally signaled by the so-called Landau pole with $\alpha_s(\Lambda_{\mathrm{QCD}}) = \infty$. We emphasise that a large or diverging coupling does *not* imply confinement, the theory could still be QEDsS-like showing a Coulomb-potential with a large coupling. The latter would not lead to the absence of colored asymptotic states but rather to so-called color charge superselection sectors as in QED. There, we have asymptotic charged states and no physics process can change the charge.

Appendix A: Feynman rules for QCD in the covariant gauge

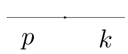
In this Appendix we depict the Feynman rules for QCD in the general covariant gauge.



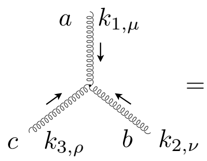
$$= \delta^{ab}\delta^{(4)}(p+k)\left(\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2}$$



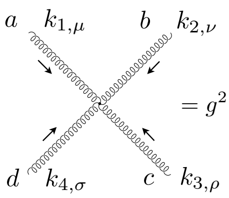
$$= -\delta^{ab}\delta^{(4)}(p+k)\frac{1}{p^2}$$



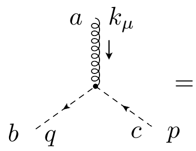
$$= \delta^{(4)}(p+k)\frac{1}{i\not{p}+m}$$



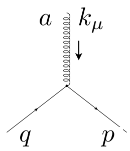
$$= i\,g f^{abc}(2\pi)^4\delta^{(4)}(k_1+k_2+k_3)\Big[(k_2-k_1)_\rho\,\delta_{\mu\nu} + (k_1-k_3)_\nu\,\delta_{\mu\rho} + (k_3-k_2)_\mu\,\delta_{\nu\rho}\Big]$$



$$= g^2(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{4}k_i\right)\Big[f^{iab}f^{icd}\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) + f^{iac}f^{ibd}\left(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) + f^{iad}f^{ibc}\left(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma}\right)\Big]$$



$$= -i\,g f^{abc}p_\mu(2\pi)^4\delta^{(4)}(p-q-k)$$



$$= -i\,g\gamma_\mu T^a\,(2\pi)^4\,\delta^{(4)}(p-q-k)$$

FIG. 23: Feynman rules.

Appendix B: Gribov copies

In Chapter I A 2 we have derived the gauge-fixed path integral under the assumption that there is only one representative of the gauge orbit that satisfies the gauge fixing condition. However, there might be several (Gribov) copies, i.e. several physically equivalent solutions to the gauge fixing condition that are related by gauge transformations not yet fixed by the gauge fixing condition $\mathcal{F}=0$. Indeed, any sufficiently smooth gauge exhibits (infinite many) Gribov copies, $\sum_{\text{Gribov copies}} = \#_{\text{Gr}}$. As for the integration over the gauge group, $\#_{\text{Gr}}$ occurs in the numerator as well as the denominator in (18) and hence cancels. It is left to compute the Jacobian $J[A] = \Delta_{\mathcal{F}}[A]$. To that end we use the representation of the Dirac δ-function

$$\delta[\mathcal{F}[A^{\mathcal{U}}]] = \sum_{i=1}^{\#_{\rm Gr}} \frac{1}{|\det \frac{\delta \mathcal{F}}{\delta \omega}|} \delta[\omega - \omega_i] \quad \text{with} \quad \mathcal{U} = e^{i\omega}. \tag{B1}$$

which leads to

$$\Delta_{\mathcal{F}}[A] = \left(\sum_{i=1}^{\#_{\rm Gr}} \frac{1}{|\det \mathcal{M}_{\mathcal{F}}[A^{e^{i\omega_i}}]|} \right)^{-1} \qquad \text{with} \qquad \mathcal{M}_{\mathcal{F}}[A] = \frac{\delta \mathcal{F}}{\delta \omega}\bigg|_{\omega=0} [A^{e^{i\omega}}]\,. \tag{B2}$$

In the QFTII lecture notes in chapter IV, Appendix A the occurance of the Gribov copies in gauge field reparameterisations due to gauge fixings is elucidated at the simple example of the reparameterisation of a two-dimensional intergal.

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