2.2 Noether theorem

'Every continuous symmetry of the action leads to a conserved current density and a conserved charge'

Symmetry transformation (e.g. $\delta_\varepsilon \phi = i\varepsilon\phi$
$\phi \to e^{i\alpha}\phi$)

$$\phi(x) \longrightarrow \phi(x) + \delta_\varepsilon \phi(x) \tag{2.22}$$

with $S \to S[\phi(x) + \delta_\varepsilon \phi(x)]\Big|_{EoM} \overset{!}{=} S[\phi(x)]$

This is satisfied for

$$\mathcal{L} \to \mathcal{L} + \varepsilon\, \partial_\mu J^\mu(\phi) \tag{2.23}$$

total derivative

$$\Rightarrow \int d^4x\, \mathcal{L} \to \int d^4x\, \mathcal{L} + \varepsilon \underbrace{\int d^4x\, \partial_\mu J^\mu(\phi)}_{0 \text{ if } J(\phi)|_{\text{infinity}} = 0} \tag{2.24}$$

It follows that

$$\mathcal{L} \rightarrow \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \delta_\varepsilon \phi + \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \partial_\nu \delta_\varepsilon \phi$$

$$= \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \delta_\varepsilon \phi + \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \delta_\varepsilon \phi \right) - \left(\partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \right) \delta_\varepsilon \phi$$

Euler-Lagrange

$$= \mathcal{L} + \partial_\nu \left(\frac{\delta \mathcal{L}}{\partial \partial_\nu \phi} \delta_\varepsilon \phi \right) + (\mathrm{EoM}) \delta_\varepsilon \phi$$

$$\overset{!}{=} \mathcal{L} + \varepsilon \partial_\nu J^\nu \qquad (2.25)$$

Evaluated at the EoM: $\left. \frac{\partial \delta_\varepsilon \phi}{\partial \varepsilon} \right|_{\varepsilon=0} = \Delta \phi$

$$\partial_\nu J^\nu = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \Delta \phi \right) \qquad (2.26)$$

or

$$\boxed{j^\nu = \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \Delta \phi - J^\nu}$$ conserved current

Noether Charge:

$$\boxed{Q(t) = \int d^3x \; j^0(t, \vec{x})}$$

$$\dot{Q}(t) = \int d^3x \; \partial_0 j^0(t, \vec{x})$$

$$= + \int d^3x \; \vec{\partial} \, \vec{j}(t, \vec{x}) = 0 \qquad (2.27)$$

Remarks

(i) Extends to general symmetry with r parameters: $\Delta_r \phi = \left.\frac{\partial \delta_\varepsilon \phi}{\partial \varepsilon_r}\right|_{\varepsilon = 0}$

$$\boxed{j_r^\nu = \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_i} \Delta_r \phi_i - \mathcal{J}_r^\nu} \qquad (2.28)$$

multiple fields

(ii)* Split of $\delta_\varepsilon \phi = \underbrace{\delta_{\phi_\varepsilon} \phi}_{\text{Trafo of } \phi} + \underbrace{\delta_\varepsilon x_\nu}_{\text{Trafo of } x} \partial^\nu \phi$

leads to, $\Delta_r x = \frac{\partial \delta_\varepsilon x}{\partial \varepsilon_r}$ (2.29)

$$j_r^\nu = \Delta_r \phi \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} + \Delta_r x^\nu \left\{ \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} - \mathcal{L} \eta^\nu{}_\nu \right\} - \mathcal{J}_r^\nu .$$

(1) Energy-momentum conservation

Symmetry: Translations

'Physics is invariant under
a shift of the lab'

$$\phi(x) \to \phi(x+a) \qquad (2.30)$$

Infinitesimally

$$\phi(x) \to \phi(x+\varepsilon) = \phi(x) + \varepsilon^{\nu} \overbrace{\partial_{\nu}\phi(x)}^{\Delta_{\nu}\phi} + \mathcal{O}(\varepsilon^2) \qquad (2.31)$$

It follows that

$$\begin{aligned} \mathcal{L}(\phi(x), \partial_{\mu}\phi(x)) &\to \mathcal{L} + \varepsilon^{\nu} \partial_{\nu} \mathcal{L}(x) + \mathcal{O}(\varepsilon^2) \\ &= \mathcal{L} + \varepsilon^{\nu} \partial_{\mu} \eta^{\mu}{}_{\nu} \mathcal{L} + \mathcal{O}(\varepsilon^2) \end{aligned} \qquad (2.32)$$

With $r=\nu$ we have $\mathcal{L} \to \mathcal{L} + \varepsilon^{r} \partial_{\mu} J^{\mu}_{r}$ with

$$J^{\mu}{}_{\nu} = \eta^{\mu}{}_{\nu} \mathcal{L} \qquad (2.33)$$

Noether current (2.28):

$$\boxed{j^{\mu}{}_{\nu} = \partial_\nu \phi \, \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \eta^{\mu}{}_{\nu} \mathcal{L} =: T^{\mu}{}_{\nu}} \qquad (2.34)$$

$T^{\mu\nu}$: energy-momentum tensor

$$\partial_\mu T^{\mu\nu} = 0 \qquad (2.35)$$

- four conserved currents / charges

$$P^{\mu} = \int d^3x \, T^{0\mu} \qquad (2.36)$$

4-momentum

energy-density

$$P^{0} = \int d^3x \, T^{00} = \partial^0 \phi \, \overbrace{\frac{\partial \mathcal{L}}{\partial \partial_0 \phi}}^{\pi} - \mathcal{L}$$

$$= \mathcal{H} \quad \text{Hamiltonian} \qquad (2.37)$$

Scalar field: $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V$

$$\mathcal{H} = (\partial_0 \phi)^2 - \mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)$$

Remarks

- Covariance of p^ν is not apparent

Note, however, that

$$\int d^3x \, \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} \sim d^3x \, dx^0 \qquad (2.38)$$

0-comp. of four-vector

d^4x is invariant! Exercise: Show covariance of P^μ.

- P^i generates translations:

$$P^i = \int d^3x \, T^{0i} = \int d^3x \, \underbrace{\frac{\partial \mathcal{L}}{\partial \partial_0 \phi}}_{\pi} \partial^i \phi$$

real scalar field (2.2), p. 4

$$\downarrow \qquad (2.39)$$

$$= \int d^3x \, \pi \, \vec{\nabla} \phi$$

Poisson bracket: $\{ P^i, \phi(\vec{x}) \} = - \vec{\nabla} \phi$

$$\text{with } \{ \phi(\vec{x}), \pi(\vec{y}) \} = \delta(\vec{x} - \vec{y}) \qquad (2.40)$$

Quantisation:

$$\{ A, B \} \rightarrow [\hat{A}, \hat{B}] \qquad (2.41)$$

- in general, $T^{\mu\nu}$ in (2.34) is

<u>not</u> symmetric due to $\partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}$,

but always can be symmetrised.

The latter property is important for the coupling to gravity.

Alternatively, one can define

$$\boxed{T^{\mu\nu}_{sym} = \frac{1}{\sqrt{-\det g}} \frac{\delta S}{\delta g^{\mu\nu}}} \qquad (2.42)$$

- * $\phi'(x') = \phi(x)$ for scalar field

(So far we have used $\phi(x) \to \phi(x')$)

$$\Rightarrow \Delta\phi = 0$$

$$\Delta_\rho x^\nu = \eta_\rho{}^\nu$$

$$J^\mu{}_\rho = 0 \qquad (\mathcal{L}' = \mathcal{L})$$

Check that $j^\mu{}_\rho$ in (2.29), p.10 is (2.39)

(2) Charge of a complex scalar field

action (2.20), p. 7 : $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi \phi^*$$

U(1)-Symmetry :

$$\phi \to \phi' = e^{i\varepsilon} \phi \quad = \phi + i\varepsilon\phi + \ldots$$

$$\sim \phi^* \to \phi^{*\prime} = \phi^* e^{-i\varepsilon} = \phi - i\varepsilon\phi + \ldots$$

(2.43)

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} \Rightarrow J^\mu = 0$$

Noether current (2.28), p. 10 , $\Delta\phi = i\phi$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \Delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} \Delta\phi^*$$

$$= \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} i\phi - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} i\phi^*$$

(2.44)

$$= i\left[\partial^\mu \phi^* \phi - \partial^\mu \phi \, \phi^*\right]$$

Noether charge (2.27), p. 10

$$Q = \int d^3x \, j^0 = i \int d^3x \left[\phi^* \partial_t \phi - \partial_t \phi^* \phi \right] \qquad (2.45)$$

Conservation law:

$$\dot{Q}\Big|_{EoM} = i \int d^3x \Big[\cancel{\dot{\phi}^* \dot{\phi} - \dot{\phi}^* \dot{\phi}} + \phi^* \partial_t^2 \phi - \partial_t \phi^* \phi \Big]$$

$$\text{EoM} \rightarrow \quad = i \int d^3x \Big[\phi^* (\Delta - m^2) \phi - (\Delta - m^2)\phi^* \phi \Big] \qquad \Delta = \vec{\nabla}^2$$

$$= i \int d^3x \left[\phi^* \Delta \phi - \Delta \phi^* \phi \right] \underset{\text{part. int.}}{=} 0 \qquad (2.46)$$

In momentum space (with (2.21), p. 7)

$$Q = \int \frac{d^3p}{(2\pi)^3} \underbrace{\frac{1}{2\omega_p}}_{\text{Normalisation of } \alpha, \beta} \Big\{ \underbrace{\alpha^*(\vec{p}) \, \alpha(\vec{p})}_{\text{part.}} - \underbrace{\beta^*(\vec{p}) \beta(\vec{p})}_{\text{anti-part.}} \Big\}$$

Quantisation: $\alpha, \beta \rightarrow$ op. a, b

(2.47)