2.3 Quantisation

(1) QFT is the field-theoretical limit of quantum mechanics. Therefore

$$[\hat{q}, \hat{p}] = i\hbar \qquad \longrightarrow \qquad [\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$$

$$[\hat{q}, \hat{q}] = 0 = [\hat{p}, \hat{p}] \qquad\qquad [\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = 0 = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})]$$

(2.48)

0+1 dim QFT $\simeq$ QM
Example: harmonic oscillator
see page 17a

Quantisation: • $\phi(x) \longrightarrow \hat{\phi}(x)$ operator

• expectation value $\langle \hat{\phi}(x) \rangle$

$\simeq$ classical field

Canonical momentum:

(2.49)

$$\boxed{\pi(t,\vec{x}) = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi}(t,\vec{x}) = \partial^0 \phi(t,\vec{x})}$$

with $\mathcal{L} = \frac{1}{2}\left\{\partial_0\phi\partial^0\phi - (\vec{\nabla}\phi)^2 - m^2\phi^2\right\}$

free real scalar field

Example: 1+0 dimensions

$$S[q] = \int dt \, \underbrace{\Big\{ \overset{(\partial_t\phi)^2}{\tfrac{1}{2}\dot{q}^2} - \tfrac{1}{2}\overset{\phi(-\vec{\nabla}^2+m^2)\phi}{\omega^2 q^2} \Big\}}_{L}$$

$$H = p \cdot \dot{q} - L$$

with $p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$

$$\Rightarrow H = \frac{1}{2} p^2 + \frac{1}{2} \cdot \omega^2 q^2$$

Quantisation: $p, q \rightarrow \hat{p}, \hat{q}$

with $[\hat{q}, \hat{p}] = i$

creation, annihilation ops.: $\boxed{\text{from now on: Drop '}\wedge\text{'}}$

$q = \frac{1}{2\omega}(a + a^+)$ usually $\frac{1}{\sqrt{2\omega}}(a+a^+)$

$p = -\frac{i}{2}(a - a^+)$ " " $-i\sqrt{\frac{\omega}{2}}(a-a^+)$

with $[a, a^+] = 2\omega$ " " $[a, a^+] = 1$

$$\Rightarrow \boxed{\hat{H} = \frac{1}{2} a^+ a + \frac{1}{2}\omega}$$

t - dependence:

Heisenberg picture: $i \frac{\partial}{\partial t} \mathcal{O}(t) = [\mathcal{O}(t), H]$

$$\Rightarrow \mathcal{O}(t) = e^{iHt} \overbrace{\mathcal{O}(0)}^{\mathcal{O}} e^{-iHt}$$

i.e.: $H(t) = H(0) = H$

It follows that

$$[q(t), p(t)] = i$$

$$= e^{iHt} \underbrace{[q, p]}_{i} e^{-iHt}$$

Hamiltonian density : $\Delta = \vec{\nabla}^2$, real scalar

$$\mathcal{H} = \Pi \cdot \partial_0 \phi - \mathcal{L} \qquad \text{partial int.} \tag{2.50}$$

$$= \frac{1}{2} \left\{ \Pi(t,\vec{x})^2 + \phi(t,\vec{x}) \left(-\Delta + m^2\right) \phi(t,\vec{x}) \right\}$$

Canonical commutation relations : $\Pi = \partial^0 \phi = \dot{\phi}$

$$\boxed{\left[\phi(t,\vec{x}), \dot{\phi}(t,\vec{y}) \right] = i\, \delta^3(\vec{x} - \vec{y})} \tag{2.51}$$

and $\left[\phi(t,\vec{x}), \phi(t,\vec{y}) \right] = 0$

see p. 17b

Remarks :

(i) ϕ satisfies EoM as does $\langle \hat{\phi} \rangle$

(ii) The free field theory describes <u>coupled</u> set of harm. osc. due to $\underline{\phi \Delta \phi}$

In Fourier space : $\phi \Delta \phi \to \phi(-p)\, \vec{p}^{\,2} \phi(p)$

$\Rightarrow \mathcal{L}, \mathcal{H}$ diagonal

$\rightsquigarrow$ Introduction of creation/annihilation ops. $a^+(\vec{p}), a(\vec{p})$ respectively

(2) Fourier space:

$\Phi(x)$ is defined as ϕ in eq. (2.16), p. 6

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \Big[a(\vec{p})\, e^{-ipx} + a^+(\vec{p})\, e^{ipx} \Big]_{p_0=\omega_{\vec{p}}}$$

operators

with $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$ (2.52a)

$$\pi(x) = \partial^0\phi(x) = \frac{-i}{2} \int \frac{d^3p}{(2\pi)^3} \Big[a(\vec{p})\, e^{-ipx} + a^+(\vec{p})\, e^{ipx} \Big]_{p_0=\omega}$$

(2.52b)

Fourier transforms:

$$\tilde{\phi}(p) := \int d^4x\, e^{ipx}\, \phi(x)$$

(2.53a)

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \tilde{\phi}(p)$$

Spatial Fourier transform ($t=0$, see p. 17a-b)

$$\boxed{\begin{aligned} \tilde{\phi}(\vec{p}) &:= \int d^3x\, e^{-i\vec{p}\vec{x}}\, \phi(\vec{x}) \\ &= \frac{1}{2\omega_{\vec{p}}} \Big[a(\vec{p}) + a^+(-\vec{p}) \Big] \end{aligned}}$$

(2.53b)

(a) Canonical commutation relations:

$$[\tilde{\phi}(\vec{p}), \tilde{\pi}(\vec{q})] = \int d^3x\, d^3y\, e^{-i(\vec{p}\vec{x}+\vec{q}\vec{y})} \underbrace{[\phi(\vec{x}), \pi(\vec{y})]}_{i\delta^3(\vec{x}-\vec{y})}$$

$$= i\int d^3x\, e^{i(\vec{p}+\vec{q})\vec{x}} = i(2\pi)^3\delta^3(\vec{p}+\vec{q}) \qquad (2.54)$$

Trivially: $[\tilde{\phi}(\vec{p}), \tilde{\phi}(\vec{q})] = 0 = [\tilde{\pi}(\vec{p}), \tilde{\phi}(\vec{q})]$

That is, $\tilde{\pi}(\vec{q})$ is conjugate to $\tilde{\phi}(-\vec{q})$.

Commutation relations for $a, a^\dagger$:

Parameterise $a, a^\dagger$ in $\tilde{\phi}, \tilde{\pi}$:

(2.52b) $\rightarrow$ $$\boxed{\tilde{\pi}(\vec{p}) = \int d^3x\, e^{-i\vec{q}\vec{x}}\partial^0\phi(\vec{x}) = -\frac{i}{2}\{a(\vec{p}) - a^\dagger(-\vec{p})\}} \qquad (2.55)$$

From (2.53b), (2.55):

$$\boxed{\begin{aligned} a(\vec{p}) &= \omega_{\vec{p}}\,\tilde{\phi}(\vec{p}) + i\,\tilde{\pi}(\vec{p}) \\ a^\dagger(-\vec{p}) &= \omega_{\vec{p}}\,\tilde{\phi}(\vec{p}) - i\,\tilde{\pi}(\vec{p}) \end{aligned}} \qquad (2.56)$$

Analogous to harmonic oscillator in QM, see p. 17a, b.

Commutator $[a, a^\dagger]$:

$$[a(\vec{p}), a^\dagger(\vec{q})] = -i\omega_{\vec{p}}\, [\tilde{\phi}(\vec{p}), \tilde{\pi}(-\vec{q})] - i\omega_{\vec{q}}\, [\tilde{\phi}(-\vec{q}), \tilde{\pi}(\vec{p})]$$

eq. (2.54) $\rightarrow$

$$= 2\omega_{\vec{p}}\, (2\pi)^3\, \delta^3(\vec{p} - \vec{q}) \qquad (2.57)$$

Also

$$[a(\vec{p}), a(\vec{q})] = 0 = [a^\dagger(\vec{p}), a^\dagger(\vec{q})] \qquad (2.58)$$

The creation and annihilation operators obey - up to the factor $2\omega_{\vec{p}}$ - canonical commutation relations of a density of harmonic oscillators.

(b) Hamiltonian density diagonal in momentum space:

$$\Delta e^{-i\vec{q}\vec{x}} = -\vec{q}^{\,2} e^{-i\vec{q}\vec{x}}$$

$$-\int d^3x\ \phi(\vec{x})\,\Delta\phi(\vec{x}) = \int d^3x \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\, e^{-i\vec{x}(\vec{p}+\vec{q})}\,\tilde{\phi}(\vec{p})\,\vec{q}^{\,2}\,\tilde{\phi}(\vec{q})$$

$$= \int \frac{d^3p}{(2\pi)^3}\ \tilde{\phi}(\vec{p})\ \vec{p}^{\,2}\ \tilde{\phi}(-\vec{p}) \qquad (2.59)$$

Analogously

$$m^2 \int d^3x\ \phi^2(\vec{x}) = m^2 \int \frac{d^3p}{(2\pi)^3}\,\tilde{\phi}(\vec{p})\,\tilde{\phi}(-\vec{p})$$

$$\int d^3x\ \pi^2(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\,\tilde{\pi}(\vec{p})\,\tilde{\pi}(-\vec{p}) \qquad (2.60)$$

$\Rightarrow$ Diagonal Hamiltonian

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{2}\left\{ \tilde{\pi}(\vec{p})\,\tilde{\pi}(-\vec{p}) + \omega_{\vec{p}}^2\,\tilde{\phi}(\vec{p})\,\tilde{\phi}(-\vec{p})\right\}} \qquad (2.61)$$

with

$$\omega_{\vec{p}}^2 = \vec{p}^{\,2} + m^2$$

Physics interpretation of Hamiltonian H is best done in terms of $a, a^\dagger$:

Use $\tilde{\phi}(\vec{p}) = \frac{1}{2\omega_{\vec{p}}}\left[a(\vec{p}) + a^\dagger(-\vec{p})\right]$ (2.53b)

$\tilde{\pi}(\vec{p}) = -\frac{i}{2}\left[a(\vec{p}) - a^\dagger(-\vec{p})\right]$ (2.55)

As in the QM-example (p. 17a-b) it follows

$$H = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\left\{\frac{1}{2}a^\dagger(\vec{p})\,a(\vec{p}) + \frac{1}{4}\left(a^\dagger(\vec{p})\,a^\dagger(-\vec{p}) + a(\vec{p})\,a(-\vec{p})\right)\right.$$

$$+ \frac{1}{2}a^\dagger(\vec{p})\,a(\vec{p}) - \frac{1}{4}\left(a^\dagger(\vec{p})\,a^\dagger(-p) + a(\vec{p})\,a(-\vec{p})\right)$$

$$\left. + \frac{1}{2}\underbrace{\left[a(\vec{p}),\, a^\dagger(\vec{p})\right]}_{2\omega_{\vec{p}}(2\pi)^3\delta^3(0)}\right\} \leftarrow \text{eq. } (2.57)$$

$$= \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\; a^\dagger(\vec{p})\,a(\vec{p}) + \frac{1}{2}V\int\frac{d^3p}{(2\pi)^3}\,\omega_{\vec{p}} \qquad (2.62)$$

with

$$(2\pi)^3\delta^3(0) = \int d^3x\, e^{i\vec{x}(\vec{p}-\vec{p})} = \underset{\text{Volume of } \mathbb{R}^3}{V}$$

Remarks:

(i) In QFT the vacuum energy density is divergent: $\frac{1}{2}\int d^3p/(2\pi)^3\ \omega_{\vec{p}}$

(ii) For QFT in a finite Volume the volume factor is finite, e.g. $\mathbb{R}^3 \to T^3, S^3$

(iii) Infinite constant will be dropped from now on; it does not play a rôle for energy differences.

It does, however, play a rôle

- at finite Temperature
- QFT coupled to gravity
  'cosmological constant problem'
- QFT with boundary conditions
  Casimir effect in QED:
  attractive force between conducting plates

Finally, we have the Hamiltonian

$$\boxed{H = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, a^\dagger(\vec{p})\, a(\vec{p})} \qquad (2.63)$$

with $\omega_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}$; $[a(\vec{p}), a^\dagger(\vec{q})] = 2\omega_{\vec{p}} (2\pi)^3 \cdot \delta(\vec{p}-\vec{q})$

H is the Hamiltonian of a momentum continuum of harmonic oscillators with frequencies $\omega_{\vec{p}}$.
Interpretation of $a, a^\dagger$ is that of annihilation and creation operators respectively.

(c) Hilbert space construction (Fock space)

(i) Vacuum (ground state) $|0\rangle$:

$$H|0\rangle = 0$$

with $\boxed{a(\vec{p})|0\rangle = 0}$ (2.64)

Normalised: $\langle 0|0\rangle = 1$

All states are generated by applying a, a^+ on the vacuum $|0\rangle$:

a, a^+ are annihilation and creation ops.

(ii) One-particle state: $\boxed{H\,|\vec{p}\rangle = \omega_{\vec{p}}\,|\vec{p}\rangle}$ p.26a

$$\boxed{|\vec{p}\rangle = a^+(\vec{p})\,|0\rangle} \qquad (2.65)$$

Orthogonality: $\langle\vec{p}| = |\vec{p}\rangle^+ = \langle 0|\,a(\vec{p})$

$$\langle\vec{p}\,'|\vec{p}\rangle = \langle 0|\,a(\vec{p}\,')\,a^+(\vec{p})|0\rangle$$

$$a|0\rangle = 0 \longrightarrow \quad = \langle 0|\left[a(\vec{p}\,'),\, a^+(\vec{p})\right]|0\rangle$$

$$= 2\,\omega_{\vec{p}}\,(2\pi)^3\,\delta^3(\vec{p}-\vec{p}\,') \qquad (2.66)$$

General one-particle state:

$$\boxed{|f\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\,f(\vec{p})\,a^+(\vec{p})\,|0\rangle} \qquad (2.67)$$

$$\Rightarrow \langle f|f\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\,f\,f^*(\vec{p})$$

properties under Lorentz trafo's $\rightarrow$ (VI)

$|\vec{p}\rangle$ is eigenstate of H_0:

$$H\,|\,\vec{p}\rangle = \frac{1}{2}\int \frac{d^3p'}{(2\pi)^3}\; a^\dagger(\vec{p}')\,a(\vec{p}')\;a^\dagger(\vec{p})\,|\,0\rangle$$

$$= \frac{1}{2}\int \frac{d^3p'}{(2\pi)^3}\; a^\dagger(\vec{p}')\,\underbrace{\left[a(\vec{p}'),\,a^\dagger(\vec{p})\right]}_{(2\pi)^3\,2\omega_{\vec{p}}\,\delta^3(\vec{p}-\vec{p}')}|\,0\rangle$$

$$= \omega_{\vec{p}}\;a^\dagger(\vec{p})\,|\,0\rangle$$

$$= \omega_{\vec{p}}\;|\,\vec{p}\rangle$$

For general one-particle state:

$$H\,|\,f\rangle = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\; f(\vec{p})\;a^\dagger(\vec{p})\,|\,0\rangle$$

N-particle state:

$$H\,|\,\vec{p}_n \ldots \vec{p}_1\rangle$$

$$= \left(\sum_{i=1}^{n} \omega_{\vec{p}_i}\right)|\,\vec{p}_n \ldots \vec{p}_1\rangle$$

(iii) N-particle states: $\boxed{H|\vec{p}_n \cdots \vec{p}_1\rangle = \left(\sum_{i=1}^{n} \omega_{\vec{p}_i}\right)|\vec{p}_n \cdots \vec{p}_1\rangle}$

p. 26a

$$a^\dagger(\vec{p}_2)\, a^\dagger(\vec{p}_1)|0\rangle = |\vec{p}_2\, \vec{p}_1\rangle$$

$$a^\dagger(\vec{p}_3)\, a^\dagger(\vec{p}_2)\, a^\dagger(\vec{p}_1)|0\rangle = |\vec{p}_3\, \vec{p}_2, \vec{p}_1\rangle$$

$$\vdots \tag{2.68}$$

- states have Bose symmetry:

$$|\vec{p}_n, \cdots \vec{p}_{i+1}\, \vec{p}_i, \cdots \vec{p}_1\rangle = |\vec{p}_n, \cdots \vec{p}_i\, \vec{p}_{i+1}, \cdots \vec{p}_1\rangle \tag{2.69}$$

as

$$a^\dagger(\vec{p}_{i+1})\, a^\dagger(\vec{p}_i) = a^\dagger(\vec{p}_i)\, a^\dagger(\vec{p}_{i+1})$$

- energy-momentum is additive:

take some state $|\beta\rangle$, then

$a^\dagger(\vec{p})|\beta\rangle$ is a state with <u>one</u>

additional particle with momentum $\vec{p}$:

Sketch: $H|\beta\rangle = E_\beta |\beta\rangle$

then
$$H\left(a^\dagger(\vec{p})|\beta\rangle\right) = a^\dagger(\vec{p})\, H|\beta\rangle + \overbrace{[H, a^\dagger(\vec{p})]}^{\omega_{\vec{p}}\, a^\dagger(\vec{p})}|\beta\rangle$$
$$= (H + \omega_{\vec{p}})\left(a^\dagger(\vec{p})|\vec{\beta}\rangle\right) \tag{2.70}$$

- annihilation:

  $a(\vec{p})|\beta\rangle$ is a state, where a particle with momentum $\vec{p}$ is removed.

Example: consider one-particle state $|f\rangle$

(2.67), p. 26

$$a(\vec{p})|f\rangle = a(\vec{p}) \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}'}} f(\vec{p}')\, a^+(\vec{p}')|0\rangle$$

$$= \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}'}} f(\vec{p}') \left[a(\vec{p}), a^+(\vec{p}')\right]|0\rangle$$

$$= f(\vec{p})|0\rangle \qquad (2.71)$$

(iv) Interpretation of $\phi(x)$

- states with defined particle number n have vanishing expectation value of ϕ, as ϕ creates and annihilates a particle.

For example: $\langle 0 | \phi(x) | 0 \rangle = 0$

with $\langle 0 | a | 0 \rangle = 0 = \langle 0 | a^\dagger | 0 \rangle$

also $\langle \vec{p} \,|\, \phi(x) \,|\, \vec{p} \rangle = 0$

$\vdots$

$\langle \vec{p}_1 \ldots \vec{p}_n \,|\, \phi(x) \,|\, \vec{p}_n \ldots \vec{p}_1 \rangle = 0$

with $|\vec{p}_n \ldots \vec{p}_1 \rangle = a^\dagger(\vec{p}_n) \ldots a^\dagger(\vec{p}_1)|0\rangle$ (2.72)

Classical field:

$$\langle \alpha | \phi(x) | \alpha \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \left\{ e^{-ipx} \alpha(\vec{p}) + e^{ipx} \alpha^*(\vec{p}) \right\} \tag{2.73}$$

with coherent state $|\alpha\rangle$:

$$a(\vec{p})|\alpha\rangle = \alpha(\vec{p})|\alpha\rangle \tag{2.74}$$

unchanged by annihilation (detection) of a particle with momentum p.

It follows that

$$|\alpha\rangle = \frac{1}{N(\alpha)} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{\infty} \left[\int \frac{d^3p_i}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}_i}} \alpha(\vec{p}_i) \right] |\vec{p}_1, \dots, \vec{p}_n\rangle \tag{2.75}$$

Normalisation

with $a(\vec{p})|\vec{p}_1 \dots \vec{p}_n\rangle$

$$= \sum_{i=1}^{n} (2\pi)^3 \, 2\omega_{\vec{p}_i} \, |\vec{p}_1 \dots, \vec{p}_{i-1} \, \vec{p}_{i+1} \dots \vec{p}_n\rangle \cdot \delta^3(\vec{p} - \vec{p}_i) \tag{2.76}$$

Normalisation $N(\alpha)$ such that

$$\langle \alpha | \alpha \rangle = 1 \tag{2.77}$$

$$\Rightarrow N(\alpha) = \exp\left\{ \tfrac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} |\alpha(\vec{p})|^2 \right\}$$

and finally

$$\boxed{|\alpha\rangle = \frac{1}{N(\alpha)} \exp\left\{ \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \alpha(\vec{p}) \, a^\dagger(\vec{p}) \right\} |0\rangle} \tag{2.78}$$

Some technical details:

From (2.76) it follows that

$$\frac{1}{n!}\, a(\vec{p}) \left[\int \frac{d^3p'}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}'}} \alpha(\vec{p}')\, a^\dagger(\vec{p}') \right]^n |0\rangle$$

$$= \frac{1}{n!} \quad \left[\int \frac{d^3p'}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}'}} \alpha(\vec{p}')\, a^\dagger(\vec{p}') \right]^{n-1} |0\rangle$$

$$= \alpha(\vec{p}) \frac{1}{(n-1)!} \left[\int \frac{d^3p'}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}'}} \alpha(\vec{p}')\, a^\dagger(\vec{p}') \right]^{n-1} |0\rangle$$

and similarly

$$\langle 0| \frac{1}{n!} \left[\int \frac{d^3p'}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}'}} \alpha^*(\vec{p}')\, a(\vec{p}') \right]^n a^\dagger(\vec{p})$$

$$= \langle 0| \frac{1}{(n-1)!} \left[\int \frac{d^3p'}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}'}} \alpha^*(\vec{p}')\, a(\vec{p}') \right]^{n-1} \alpha^*(\vec{p})$$

with $a(\vec{p})\, a^\dagger(\vec{p}_1) \cdots a^\dagger(\vec{p}_n)\, |0\rangle$

$$= \Big(\underbrace{\left[a(\vec{p}), a^\dagger(\vec{p}_1) \right]}_{(2\pi)^3 2\omega_{\vec{p}}\, \delta(\vec{p}-\vec{p}_1)} + a^\dagger(\vec{p}_1) \cdot a(\vec{p}) \Big)\, a^\dagger(\vec{p}_2) \cdots a^\dagger(\vec{p}_n) |0\rangle$$

$$= (2\pi)^3 2\omega_{\vec{p}}\, \delta(\vec{p}-\vec{p}_1)\, a^\dagger(\vec{p})\, a^\dagger(\vec{p}_2) \cdots a^\dagger(\vec{p}_n)\, |0\rangle$$
$$a^\dagger(\vec{p}_1)\, a(\vec{p})\, a^\dagger(\vec{p}_2) \cdots a^\dagger(\vec{p}_n)\, |0\rangle$$

$$\omega_i = \omega(\vec{k}_i) = \sqrt{\vec{k}_i^{\,2} + m^2}\,, \quad \omega_{i'} = \omega(\vec{k}_i')$$

$$\langle \alpha | \alpha \rangle = \frac{1}{N^2(\alpha)} \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right)^2 \int \prod_{i=1}^{n} \frac{d^3k_i}{(2\pi)^3} \frac{d^3k_i'}{(2\pi)^3} \frac{1}{2\omega_i} \frac{1}{2\omega_{i'}}$$

$$\left. \alpha^*(\vec{k}_i)\, \alpha(\vec{k}_i') \right] \langle 0 | a(\vec{k}_1) \circ\circ\circ a(\vec{k}_n)\, a^\dagger(\vec{k}_n') \circ\circ\circ a(\vec{k}_1') | 0 \rangle$$

—

$$\langle 0 | a(\vec{k}_1) \circ\circ\circ \underbrace{a(\vec{k}_n) \cdot a^\dagger(\vec{k}_n')}_{\underbrace{[a(\vec{k}_n), a^\dagger(\vec{k}_n')]}_{(2\pi)^3 2\omega_n \delta(\vec{k}_n - \vec{k}_n')} + a^\dagger(\vec{k}_n')\, a(\vec{k}_n)} \circ\circ\circ a(\vec{k}_1') | 0 \rangle$$

$$= (2\pi)^3 2\omega_n \delta(\vec{k}_n - \vec{k}_n') \langle 0 | a(\vec{k}_1) \circ\circ\circ a(\vec{k}_{n-1}) \circ a^\dagger(\vec{k}_{n-1}') \circ\circ\circ a^\dagger(\vec{k}_1') | 0 \rangle$$

$$+ \langle 0 | a(\vec{k}_1) \circ\circ\circ a(\vec{k}_{n-1})\, a^\dagger(\vec{k}_n') \underbrace{a(\vec{k}_n)\, a^\dagger(k_{n-1}')}_{[a(\vec{k}_n), a^\dagger(\vec{k}_{n-1}')] + a^\dagger(\vec{k}_{n-1}') \cdot a(\vec{k}_n)} \circ\circ\circ a^\dagger(\vec{k}_1') | 0 \rangle$$

$$[a(\vec{k}_n), a^\dagger(\vec{k}_{n-1}')] = (2\pi)^3 2\omega_n \delta(\vec{k}_n - \vec{k}_{n-1}')$$

$$\vdots$$

$$= (2\pi)^3 2\omega_n \sum_{i=1}^{n} \langle 0 | a(\vec{k}_1) \cdot\cdot\cdot a(\vec{k}_{n-1})\, a^\dagger(\vec{k}_n') \circ\circ\circ \widehat{a^\dagger(\vec{k}_i')} \circ\circ\circ a^\dagger(\vec{k}_1') | 0 \rangle$$

$$\cdot (2\pi)^3 2\omega_n \delta(\vec{k}_n - \vec{k}_i')$$

where $\widehat{a^\dagger(\vec{k}_i')} = 1$

$$\Rightarrow \langle \alpha | \alpha \rangle = \frac{1}{N^2} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(n-1)!} \int \frac{d^3k_n}{(2\pi)^3} \frac{1}{2\omega_n} \alpha^*(\vec{k}_n) \alpha(\vec{k}_n)$$

$$\int \prod_{i=1}^{n-1} \left[\frac{d^3k_i}{(2\pi)^3} \frac{d^3k_i'}{(2\pi)^3} \frac{1}{2\omega_i} \frac{1}{2\omega_i'} \right] \langle 0 | a(\vec{k}_1) \cdots a(\vec{k}_{n-1}) a^+(\vec{k}'_{n-1}) \cdots a^+(\vec{k}'_1) | 0 \rangle$$

$$\cdot \alpha^*(\vec{k}_i) \alpha(k_i')$$

$$\vdots$$

$$= \frac{1}{N^2} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \alpha^* \alpha(\vec{k}) \right]^n$$

$$\frac{1}{N^2} e^{\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} |\alpha(\vec{k})|^2}$$

$$\Rightarrow \boxed{N = e^{1/2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} |\alpha(\vec{k})|^2}}$$

Scalar product :

$$\langle \alpha' | \alpha \rangle = \exp \left\{ -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \left(|\alpha(\vec{p})|^2 + |\alpha'(\vec{p})|^2 - 2\alpha'^{*}\alpha(\vec{p}) \right) \right\} \qquad (2.79)$$

not orthogonal.

Completeness : (1+0 dim)

$$\frac{1}{\pi} \int d^2\alpha \, |\alpha\rangle\langle\alpha| = \mathbb{1} \qquad (2.80)$$

(v) Conserved energy-momentum tensor $\partial_\nu T^{\mu\nu} = 0$

$$\underline{P}^0 = H$$

$$\boxed{\begin{aligned} \underline{\vec{P}} &= \int d^3x \, \pi \, \vec{\nabla} \phi \\ &\simeq \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \, \vec{p} \, a^{+}(\vec{p}) \, a(\vec{p}) \end{aligned}} \qquad (2.90)$$

with

$$\boxed{\underline{\vec{P}} | \vec{p} \rangle = \vec{p} \, | \vec{p} \rangle} \qquad (2.91)$$

(vi) Lorentz symmetry:

$$\mathcal{U}(\Lambda)|0\rangle = |0\rangle \qquad (2.92)$$

(unitary) Fock space representation of Lorentz trafo Λ

one-particle state:

$$|\vec{p}\rangle = a^{\dagger}(\vec{p})|0\rangle \qquad (2.93)$$

with $\langle\vec{q}|\vec{p}\rangle = \underbrace{2\omega_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{q})}_{\text{Lorentz invariant}}$

and $\mathcal{U}(\Lambda)|\vec{p}\rangle = |\Lambda\vec{p}\rangle$

Lorentz invariance of $2\omega_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{q})$:

Use Lorentz-invariant measure

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}} = \int\frac{d^4p}{(2\pi)^4}2\pi\,\delta(p^2-m^2)\,\Theta(p^0) \qquad (2.93)$$

invariant under proper, orthochronous Lorentz-trafos
$\det\Lambda=1$, $(\Lambda^0{}_0>0)$ $\quad SO(1,3)$

$$\Rightarrow \underbrace{\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}}_{\text{invariant}} \longrightarrow \underbrace{2\omega_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{q})}_{\text{invariant}} \longleftarrow \underset{\text{invariant}}{1} \qquad (2.94)$$

Points (i) – (vi) complete Fock space construction

Remarks:

(i) $\phi(x)$ generates a superposition of one particle states from the vacuum,

$$\langle 0|\phi(x)|\vec{p}\rangle = \langle 0| \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-ip'x} a(\vec{p}\,') a^\dagger(\vec{p})|0\rangle$$

$$= e^{-ipx} \qquad (2.95)$$

reminiscent of non-relativistic QM
$\langle x|p\rangle = e^{-ipx}$

(ii) Causality encoded in propagator

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

and its variants: Retarded, advanced time-ordered props → chapter 3
perturb. theory

(3) Complex scalar field

$$S[\phi] = \int d^4x \left\{ \partial_\nu \phi \, \partial^\nu \phi^* - m^2 \phi \phi^* \right\} \tag{2.96}$$

with $\phi = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2)$

Field operator:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \left\{ a(\vec{p}) e^{-ipx} + b^+(\vec{p}) e^{ipx} \right\} \tag{2.97}$$

Canonical momentum: $\pi = \partial^0 \phi^*$

$$\pi(x) = \frac{-i}{2} \int \frac{d^3p}{(2\pi)^3} \left\{ b(\vec{p}) e^{-ipx} - a^+(\vec{p}) e^{ipx} \right\} \tag{2.98}$$

with $[\phi(\vec{x}), \pi(\vec{y})] = \delta^3(\vec{x} - \vec{y})$

and

$$[a(\vec{p}), a^+(\vec{q})] = 2\omega_{\vec{p}} (2\pi)^3 \delta^3(\vec{p} - \vec{q})$$

$$[b(\vec{p}), b^+(\vec{q})] = 2\omega_{\vec{p}} (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \tag{2.99}$$

others vanish!

see Exercise

Hamiltonian:

$$H = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left\{ \tilde{\pi}(\vec{p})\,\tilde{\pi}^*(\vec{p}) + \omega_{\vec{p}}^2\,\tilde{\phi}(\vec{p})\,\tilde{\phi}^*(\vec{p}) \right\}$$

$$\leadsto \quad H_{op} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left\{ a^+(\vec{p})\,a(\vec{p}) + b^+(\vec{p})\,b(\vec{p}) \right\}$$

sum of energy of part. and anti-part. (2.100)

Charge: see p. 16 eqs (2.45), (2.47)

$$Q = i\int d^3x \left\{ \phi^*\partial_t\phi - \partial_t\phi^*\phi \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\left\{ a^+(\vec{p})\,a(\vec{p}) - b^+(\vec{p})\,b(\vec{p}) \right\}$$

charge of anti-part. is minus the charge of part. (2.101)

and

$$\langle\alpha|Q|\alpha\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\left\{ \alpha^*\alpha(\vec{p}) - \beta^*\beta(\vec{p}) \right\}$$

(2.102)