## 3.5 LSZ - formalism

(Lehmann, Symanzik, Zimmermann)

We have seen that the naive preparation of our in-state lead to a factor

$(\text{—})^{-1}$ —○— $\neq 1$ in our scattering amplitudes.

We have encountered a similar problem with vacuum bubbles before.

We shall see that

$$\phi_H(t\to\mp\infty) \to Z^{1/2}\,\phi^0_{in/out} \quad \text{(weak op. equivalence)}$$

with $Z \leq 1$. So far, we have implicitly assumed $Z=1$.

For determining $Z$, we carefully compute

$$\langle\Omega|\phi_H(x)\,\phi_H(y)|\Omega\rangle = \langle\phi(x)\,\phi(y)\rangle$$

$$= \sum\!\!\!\!\!\!\int_{\lambda,\vec{p}} \langle\Omega|\phi(x)|\lambda,\vec{p}\rangle_H\,{}_H\langle\lambda,\vec{p}|\phi(y)|\Omega\rangle \qquad (3.98)$$

$\vec{p}$ ← boosts

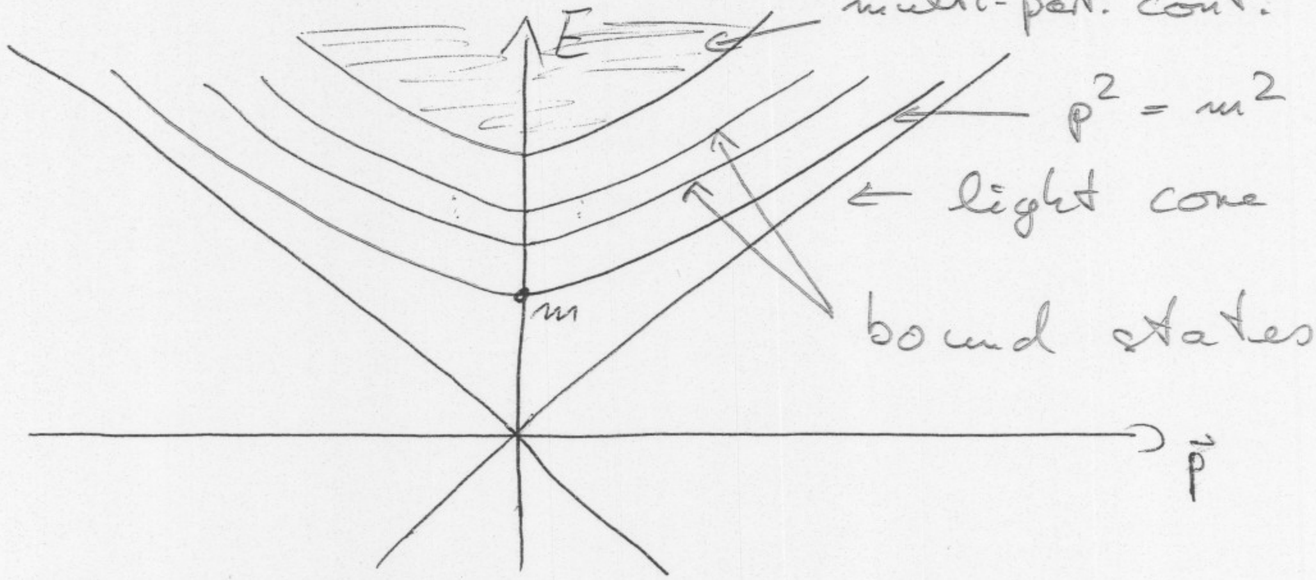
(i) Here $|\lambda\rangle$ are the eigenstates of

$H$ : $H|\lambda\rangle = E_\lambda |\lambda\rangle$

and $\vec{P}|\lambda\rangle = \vec{p}_\lambda |\lambda\rangle$

with $E_\lambda^2 - \vec{p}_\lambda^{\,2} = m_\lambda^2$ fixed. The states

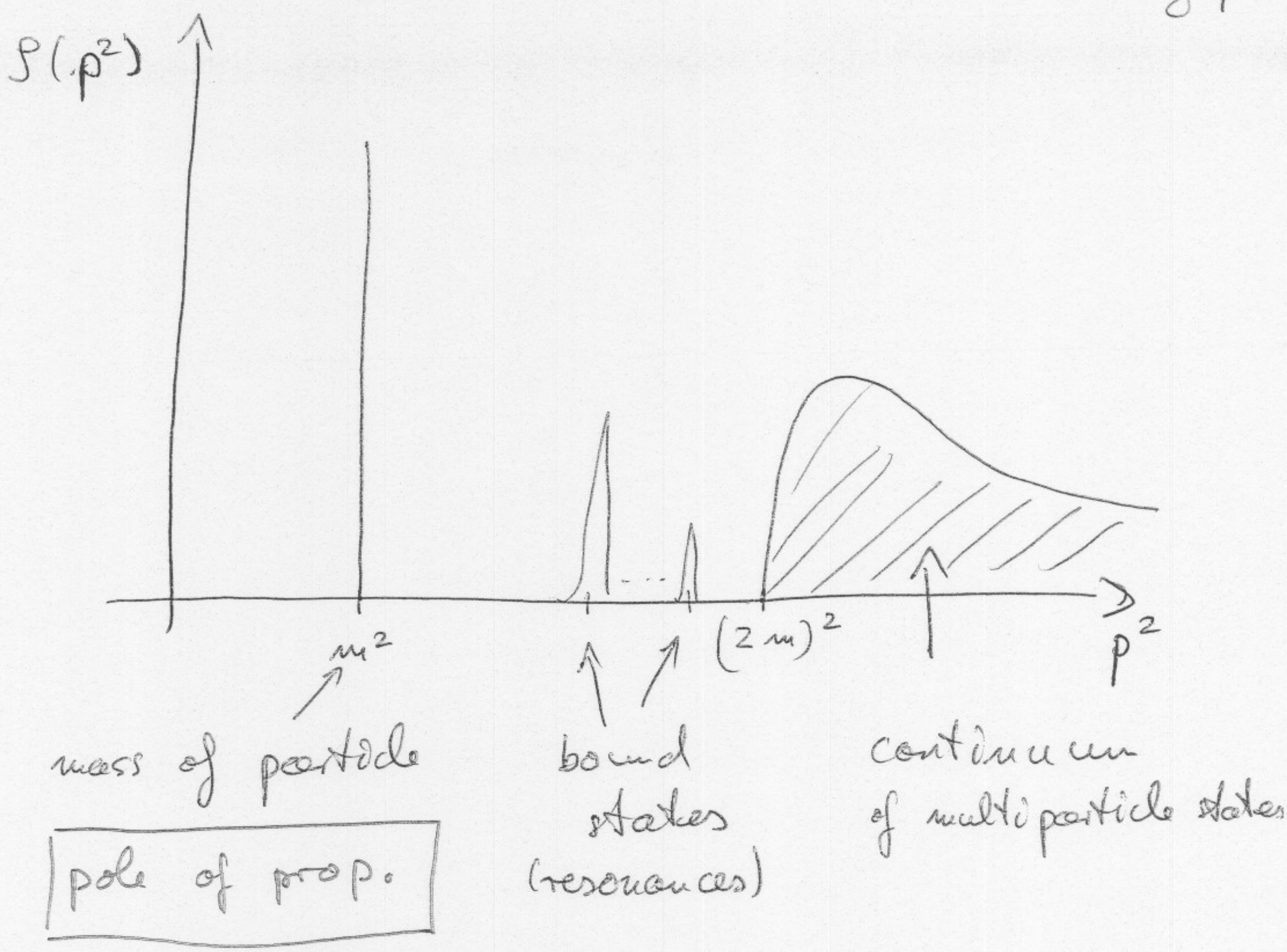
$|\lambda\rangle$ with fixed $m_\lambda$ are connected by boosts.




ii) $E_\lambda = 0 : |\Omega\rangle$

(iii) $$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_\lambda(\vec{p})} |\lambda, \vec{p}\rangle$$

(iv) $\langle\Omega|\phi(x)|\Omega\rangle = 0$




$\rho$ has the representation

$$\rho(p^2) = Z \cdot 2\pi\, \delta(p^2 - m^2) + \Theta(p^2 - \underbrace{m^2_{\text{first-res}}}_{= m_1^2}) \cdots \tag{3.103}$$

and hence

$$\langle T\phi(x)\phi(y)\rangle = Z\, D_F(x-y; m^2) + \int_{m_1^2}^{\infty} \frac{dM^2}{(2\pi)}\, \rho(M^2)\, D_F(x-y; M^2) \tag{3.104}$$

carries one-particle pole of $\phi$

Relation to $\phi_{in}$ on p. 84:

Consider one-particle states $|\lambda_1\rangle$ in $|\lambda\rangle\langle\lambda|$:

$$\rho \sim \sum_{\substack{\text{one-particle} \\ \text{states } \lambda_1}} e^{-ip_\lambda(x-y)} |\langle\Omega|\phi(0)|\lambda_1\rangle|^2$$

with, $\mathcal{U} = \mathcal{U}(-\infty, 0)$,

$$|\langle\Omega|\phi(0)|\lambda_1\rangle|^2 = |\langle\Omega|\mathcal{U}^{-1}\mathcal{U}\phi\mathcal{U}^{-1}\mathcal{U}\;|\lambda_1\rangle|^2$$

$$= |{}_I\langle 0|Z^{1/2}\phi_{in}\;|\;\lambda_1\rangle_I|^2$$

$$= Z$$

We use that with $\hat{P} = (H, \vec{P})$ we have

$$\phi(x) = e^{i\hat{P}\vec{x}}\, \phi(0)\, e^{-i\hat{P}x} \qquad (3.99)$$

Then we get, $(x^0 \geq y^0)$

$$\langle \phi(x)\phi(y) \rangle = \sum_\lambda \overbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\lambda(\vec{p})}}^{\int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m_\lambda^2)\Theta(p_\lambda^0)} |\langle \Omega|\phi(0)|\lambda,\vec{p}\rangle|^2 \cdot e^{-ip_\lambda(x-y)}$$

$$= \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\, \frac{1}{p^2-m_\lambda^2+i\varepsilon}\, e^{-ip(x-y)} \cdot \underbrace{|\langle \Omega|\phi(0)|\lambda,\vec{p}\rangle|^2}_{|\langle \Omega|\phi(0)|\lambda\rangle|^2 \;\leftarrow\; |\lambda\rangle = |\lambda,0\rangle} \qquad (3.100)$$

In summary, by using the same steps for $x^0 \leq y^0$, (Källén-Lehmann)

$$\boxed{\langle T\phi(x)\phi(y)\rangle = \int_0^\infty \frac{dM^2}{2\pi}\, \rho(M^2)\, D_F(x-y; M^2)} \qquad (3.101)$$

with spectral function

$$\rho(p^2) = \sum_\lambda (2\pi)\, \delta(p^2-m_\lambda^2)\, |\langle \Omega|\phi(0)|\lambda\rangle|^2 \qquad (3.102)$$

Determination of Z:

Consider now $<\phi(x)\,\phi(y)>$, no time-ord.!

Then $D_F$ in (3.104) is substituted by D.

Note also that

$$\left[\frac{\partial}{\partial y^0}<[\phi(x),\phi(y)]>\right]_{x^0=y^0} = \left<[\phi(x),\,\Pi(y)]_{x^0=y^0}\right>$$

$$= i\,\delta^3(\vec{x}-\vec{y}) \qquad (3.105)$$

and $\left[\frac{\partial}{\partial y^0}\big(D(x-y)-D(y-x)\big)\right]_{x^0=y^0} = i\,\delta^3(\vec{x}-\vec{y})$

Evaluate $\int d^3x <[\phi(x),\Pi(y)]_{x^0=y^0}$ with eqs. (3.104), (3.105)

$$\boxed{1 = Z + \int_{m_1^2}^{\infty} \frac{d\mu^2}{(2\pi)}\,\underbrace{\rho(\mu^2)}_{\geq 0}} \qquad (3.106)$$

Eq. (3.107) entails

$$\boxed{0 \leq Z \leq 1} \qquad (3.107)$$

Remarks:

(i) $1-Z$ accounts for overlap of $\phi|\Omega\rangle$ with multi-particle states

(ii) $Z=1$ in free theory

$Z<1$ in interacting theory

(iii) limit $t \to \mp\infty$:

$$\phi(x) \longrightarrow Z^{1/2}\phi_{in/out} \quad \text{weak op.}$$

(iv) Propagator on-shell:

$$D_F(p^2 \to m^2) \simeq \frac{iZ}{p^2 - m^2 + i\varepsilon} \qquad (3.108)$$

$$\left( = \int d^4x \, e^{ipx} \langle T\phi(x)\phi(0)\rangle \right)$$

Note: $m^2$ <u>not</u> simply mass-parameter $m_0^2$ in Lagrangian

LSZ-reduction formula:

We now extend the analysis of the two-point function to an n-point function. The latter will be related to S-matrix elements.

As in eq. (3.108) we evaluate the Fourier transform

$$\int d^4x\, e^{ipx} \langle T\,\phi(x)\,\phi(x_2)\cdots\phi(x_n)\rangle \tag{3.109}$$

With $T_+ > x_2^0, \ldots, x_n^0$, $T_- < x_2^0, \ldots, x_n^0$ we split

$$\int dx^0\, e^{ip^0x^0} = \Bigg(\underbrace{\int_{-\infty}^{T_-}}_{\text{pole}} + \underbrace{\int_{T_-}^{T_+}}_{\text{finite}} + \underbrace{\int_{T_+}^{\infty}}_{\text{pole}}\Bigg) dx^0\, e^{ip^0x^0}$$

p on-shell: $p^2 = m^2$

It follows

$$\int d^4x\, e^{ipx} \langle T\,\phi(x)\,\phi_2 \cdots \phi_n \rangle$$

$$= \int_{T_+}^{\infty} d^4x\, e^{ipx} \langle \phi(x)\, T\phi_2 \cdots \phi_n \rangle + \left( \int_{T_-}^{T_+} + \int_{-\infty}^{T_-} \right) \cdots$$

$$= \int_{T_+}^{\infty} d^4x\, e^{ipx} \sum_\lambda \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_{\vec q}} \langle \phi(x) | \lambda, \vec q \rangle \langle \lambda, \vec q | T\phi_2 \cdots \phi_n \rangle$$

$$+ \cdots \qquad (3.110)$$

We use $\langle \phi(x) | \lambda, \vec q \rangle = \langle \Omega | \phi(0) | \lambda \rangle e^{-iqx}$ :

$$\sum_\lambda \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_{\vec q}} dx^0 e^{i(p^0 - q^0 + i\varepsilon)x^0} \langle \Omega | \phi(0) | \lambda \rangle \langle \lambda, \vec p | T\phi_2 \cdots \phi_n \rangle \overbrace{(2\pi)^3 \delta^3(\vec p - \vec q)}^{\int d^3x\, e^{-i(\vec p - \vec q)\vec x}}$$

$$= \sum_\lambda \frac{1}{2\omega_{\vec p}} \frac{i\, e^{i(p^0 - \omega_{\vec p} + i\varepsilon) T_+}}{p^0 - \omega_{\vec p} + i\varepsilon} \langle \Omega | \phi(0) | \lambda \rangle \langle \lambda, \vec p | \phi_2 \cdots \phi_n \rangle \qquad (3.111)$$

For $p^0 \to \omega_{\vec p}$ : (using Källén-Lehmann)

$$\lim_{p^0 \to \omega_{\vec p}} \int_{T_+}^{\infty} d^4x\, e^{ipx} \langle T\,\phi(x) \cdots \phi(x_n) \rangle$$

$$= \frac{i\, Z^{1/2}}{p^2 - m^2 + i\varepsilon} \langle \vec p | T\phi_2 \cdots \phi_n \rangle \qquad (3.112)$$

$$+ \text{finite}$$

Evaluation of $\int_{-\infty}^{T_-}$ - term:

$$\lim_{p^0 \to -\omega_{\vec{p}}} \int_{-\infty}^{T_-} d^4x \, e^{ipx} \langle (T \phi_2 \cdots \phi_n) \phi(x) \rangle$$

$$= \frac{i Z^{1/2}}{p^2 - m^2 + i\varepsilon} \langle T \phi_2 \cdots \phi_n | -\vec{p} \rangle \quad (3.113)$$

$$+ \text{finite}$$

The last term $\int_{T_-}^{T_+} \cdots$ is finite as the integration interval has a finite length (compact)

Remarks:

(i) The above analysis can be repeated iteratively for all $\phi(x_i)$

(ii) Strictly speaking one should separate the fields specially:

$$\int d^4x \, e^{ipx} \to \int \frac{d^3k}{(2\pi)^3} e^{ipx} f_{\vec{p}}(\vec{k})$$

(iii) states $|\vec{p}\rangle$ are at time $t \to -\infty$,

" " $\langle \vec{p}|$ are at time $t \to +\infty$.

and

$${}_{-\infty}\langle \vec{p}_1 \cdots \vec{p}_n | \vec{k}_1 \cdots \vec{k}_m \rangle_{+\infty}$$

$$= \langle \vec{p}_1 \cdots \vec{p}_n | S | \vec{k}_1 \cdots \vec{k}_m \rangle \qquad (3.114)$$

$\Rightarrow$

LSZ Reduction formula:

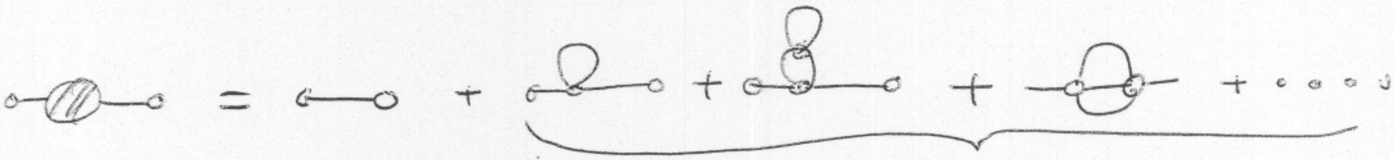
$$\boxed{\begin{aligned} &\langle \vec{p}_1 \cdots \vec{p}_n | S | \vec{k}_1 \cdots \vec{k}_m \rangle \big|_{\text{on-shell}} \\ &= \int \prod_{i=1}^{n} d^4x_i \, e^{ip_i x_i} \prod_{j=1}^{m} d^4y_j \, e^{-ik_j y_j} \\ &\quad \prod_{i=1}^{n} \left(\partial^2_{x_i} + m^2\right) \prod_{j=1}^{m} \left(\partial^2_{y_j} + m^2\right) \qquad (3.115) \\ &\quad \cdot Z^{-(n+m)/2} \langle T\, \phi(x_1) \cdots \phi(x_n) \phi(y_1) \cdots \phi(y_m) \rangle \end{aligned}}$$

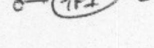
on-shell : $p^2 = m^2 \leftarrow$ physical mass pole

$m_0^2 \leftarrow$ parameter in Lagr.

Structure of $\langle T \phi_1 \cdots \phi_n \rangle$:

Example $n=2$:



$\Pi(p)$:  1PI : one-particle irreducible,

cannot be split by cutting one line

$$= \frac{i}{p^2 - m_0^2 + i\varepsilon} + \frac{i}{p^2 - m_0^2 + i\varepsilon}\left(-i\Pi(p)\right)\frac{i}{p^2 - m_0^2 + i\varepsilon}$$

$$+ \cdots$$

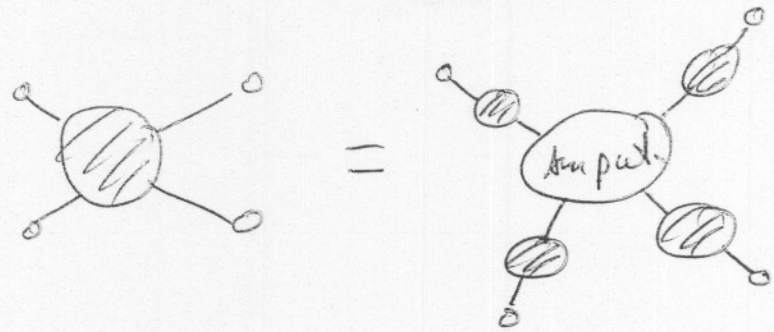
$$= \frac{i}{p^2 - \left(m_0^2 + \Pi(p)\right) + i\varepsilon} \qquad (3.116)$$

with

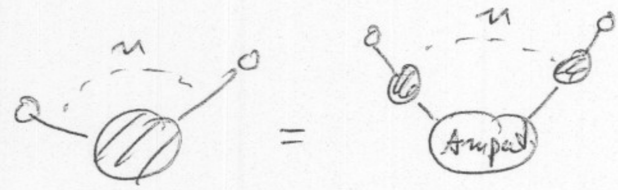
$$\frac{i}{p^2 - \left(m_0^2 + \Pi(p)\right) + i\varepsilon} \xrightarrow{p^2 \to m^2} \frac{iZ}{p^2 - m^2 + i\varepsilon}$$

(3.117)

$n = 4$:

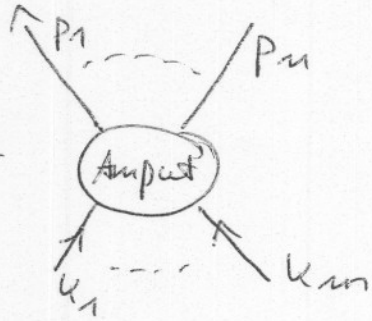



in general:




This entails for S-matrix elements with (3.115):

$$\boxed{\langle \vec{p}_1 \cdots \vec{p}_n | S | \vec{k}_1 \cdots \vec{k}_m \rangle \Big|_{\text{on-shell}} = Z^{(n+m)/2}}$$



Remarks:

(i) $Z$ is called wave function (or field strength) renormalisation, as it multiplies the field.

Note that

$$\langle T Z^{-1/2}\phi(x)\, Z^{-1/2}\phi(y)\rangle_{p^2\to m^2} = D_F(x-y; m^2)$$

$Z$ re-normalises the field

(ii) With (i) we see that

$$Z^{(n+m)/2} \langle T\, \phi(p_1) \cdots \phi(k_m)\rangle_{amp.}$$

$$\simeq Z^{(n+m)/2} \prod_i \frac{p_i^2+m^2}{Z^{1/2}} \prod_j \frac{k_j^2+m^2}{Z^{1/2}} \langle T\, \phi(p_1)\cdots\phi(k_m)\rangle$$

$$= \prod_i (p_i^2+m^2) \prod_j (k_j^2+m^2) \underbrace{\langle T\; Z^{-1/2}\phi(p_1) \cdots Z^{-1/2}\phi(k_m)\rangle}_{\text{expect. value of re-normalised fields}}$$