# 3 Perturbation theory

## 3.1 Interaction picture

Fock space construction in the previous chapter 2.3 (c) in Heisenberg picture

$$
\begin{aligned}
i\partial_t |f\rangle &= 0 \\
i\partial_t \mathcal{O}(t) &= [\mathcal{O}(t), H]
\end{aligned}
\tag{3.1}
$$

with $\mathcal{O}(t) = e^{iHt}\,\mathcal{O}\,e^{-iHt}$

The field operator $\phi(x)$ indeed follows from $\phi(\vec{x})$ by $\phi(x) = e^{iHt}\phi(\vec{x})\,e^{-iHt}$:

eal scalar: $$\phi(x) = e^{iHt}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\left\{a(\vec{p})\,e^{i\vec{p}\vec{x}} + a^\dagger(\vec{p})\,e^{-i\vec{p}\vec{x}}\right\}e^{-iHt}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\left\{a(\vec{p})\,e^{-i\omega_{\vec{p}}t+i\vec{p}\vec{x}} + a^\dagger(\vec{p})\,e^{i\omega_{\vec{p}}t-i\vec{p}\vec{x}}\right\} \tag{3.2}$$

with $$\boxed{e^{iHt}a(\vec{p})\,e^{-iHt} = a(\vec{p})\,e^{-i\omega_{\vec{p}}t}} \tag{3.3}$$

Eq. (3.3) follows with

$$H\, a(\vec{p}) = a(\vec{p})\left(H - \omega_{\vec{p}}\right) \qquad (3.4)$$

with $H = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\, a^\dagger(\vec{p})\, a(\vec{p})$

$$\Rightarrow e^{iHt}\, a(\vec{p})\, e^{-iHt} = a(\vec{p})\, e^{i(H-\omega_{\vec{p}})t}\, e^{-iHt}$$

$$= a(\vec{p})\, e^{-i\omega_{\vec{p}}t} \qquad (3.5)$$

Similarily: $e^{iHt}\, a^\dagger(\vec{p})\, e^{-iHt} = a^\dagger(\vec{p})\, e^{i\omega_{\vec{p}}t}$

Schrödinger picture:

$$i\partial_t\, |f(t)\rangle = H|f\rangle$$

$$i\partial_t\, \mathcal{O} = 0$$

with $|f(t)\rangle = e^{-iHt}|f\rangle$

Time evolution op. $\mathcal{U}(t,t') := e^{-iH(t-t')}$

acting either on states (Schrödinger) or

on operators (Heisenberg)

Remark: Causality is encoded in the op. $\phi(x)$:

$$[\phi(x), \phi(y)] = 0 \quad \text{for } (x-y)^2 < 0$$

(space-like)

(3.6)

This follows with, $\phi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \{ a(\vec{p}) e^{-ipx} + a^\dagger(\vec{p}) e^{ipx} \}$

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{\omega_{\vec{p}}} \frac{1}{\omega_{\vec{q}}} \Big\{ [a(\vec{p}), a^\dagger(\vec{q})] e^{-ipx+iqx}$$

$$+ [a^\dagger(\vec{p}), a(\vec{q})] e^{ipx-iqx} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-ip(x-y)} - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{ip(x-y)}$$

eq. 2.93, p. 32 $\longrightarrow$

$$= \int \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2) \Theta(p^0) e^{-ip(x-y)} \tag{3.7}$$

$$- \int \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2) \Theta(p^0) e^{ip(x-y)}$$

$$\Rightarrow \quad \boxed{[\phi(x), \phi(y)] = 0 \quad \text{for } (x-y)^2 < 0} \tag{3.8}$$

Proof of (3.8):

$$\int \frac{d^4p}{(2\pi)^3}\, \delta(p^2-m^2)\,\Theta(p^0)\, e^{ip(x-y)} \tag{3.9}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, \delta(p^2-m^2)\,\Theta(p^0)\, e^{ip\Lambda(x-y)}$$

with $\Lambda \in SO(1,3)$ : $p^2>0$ & $p^0>0$ : $(\Lambda p)^0>0$

For space-like $x-y$ there exist $\Lambda \in SO(1,3)$

with

$$\Lambda(x-y) = -x-y$$

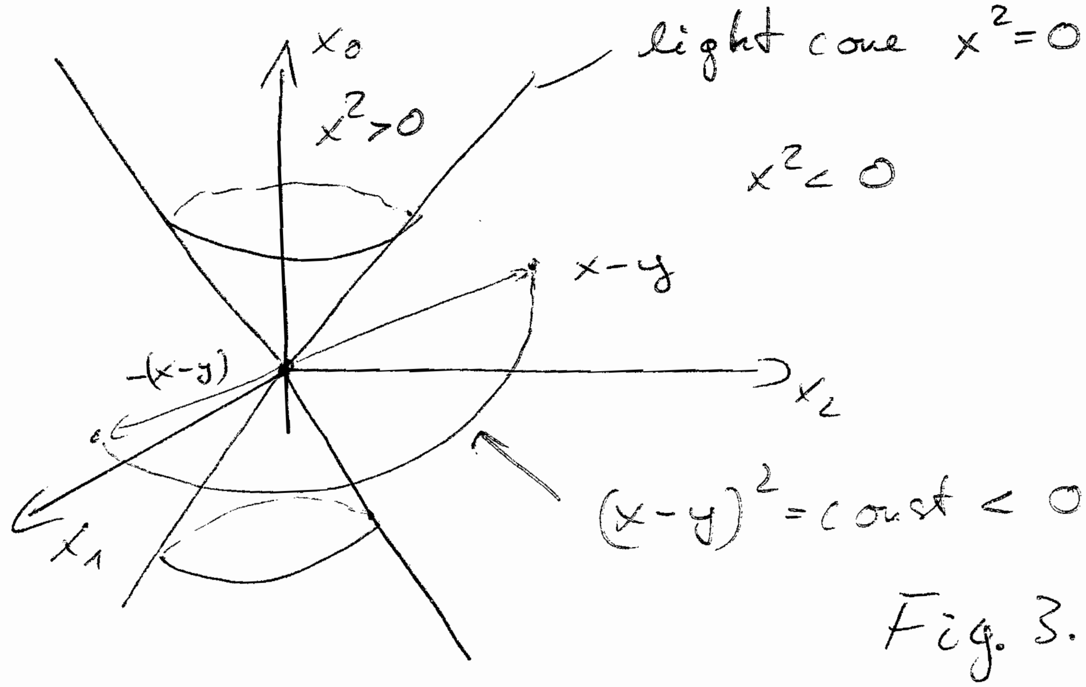


Fig. 3.1

Note that for $(x-y)^2>0$ there exist

<u>no</u> $\Lambda \in SO(1,3)$ with $\Lambda(x-y) = -(x-y)$!

Proof: see Fig. 3.1

Lagrangian density:

$$\mathcal{L}(\phi) = \underbrace{\frac{1}{2}\phi(x)(-\partial^2 - m^2)\phi(x)}_{\mathcal{L}_0(\phi)} + \underbrace{\mathcal{L}_{int}(\phi)}_{\text{interaction}} \qquad (3.10)$$

here $\quad \mathcal{L}_{int}(\phi) = -V(\phi) \quad$ polynomial in $\phi$

$\Rightarrow$ Hamiltonian density:

$$\mathcal{H}(\pi,\phi) = \overbrace{\frac{1}{2}\pi(x)^2 + \frac{1}{2}m^2\phi^2}^{\mathcal{H}_0} + \mathcal{H}_{int}(\phi) \qquad (3.11)$$

with $\quad \mathcal{H}_{int}(\phi) = V(\phi)$

Hamiltonian bounded from below:

$$\boxed{V(\phi) = \lambda/4!\ \phi(x)^4} \qquad (3.12)$$

- higher terms excluded by renormalisability (in 4d)
- $\phi^3$ term spoils symmetry $\phi \to -\phi$
- $\phi^4$-theory is 'working horse' of QFT

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Standard method in QFT: perturbation theory

- consider $\lambda \ll 1$ and expand observables, e.g. scattering amplitudes, in orders of $\lambda$: interaction is perturbation of free case!

$\Rightarrow$ Interaction picture:

- operators $O(t)$ evolve in time with free Hamiltonian $H_0 = \int d^3x\, \mathcal{H}_0$

$$i\partial_t O = [O, H_0] \Rightarrow O(t) = e^{iH_0 t} O e^{-iH_0 t} \tag{3.13}$$

- states $|f\rangle$ evolve with

$$i\partial_t |f\rangle = H_{int} |f\rangle \tag{3.14}$$

Note that $[H_0, H_{int}] \neq 0$!

$\Rightarrow \partial_t H_{int} \neq 0$ : $H_{int} = H_{int}(t)$

Hence we have

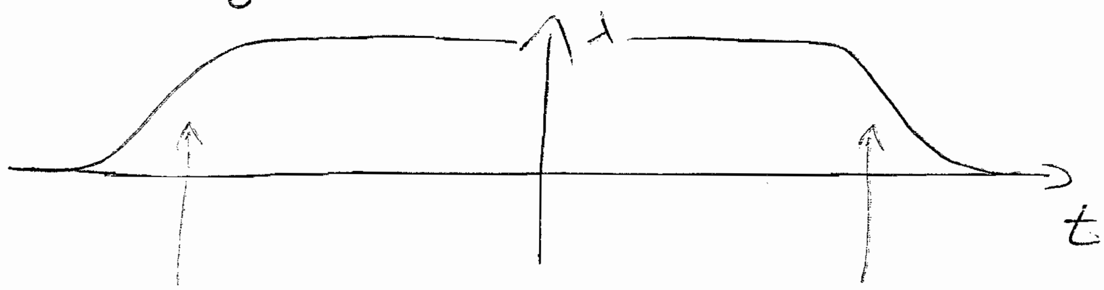
$$| f(t) \rangle = \mathcal{U}(t, t_0) | f(t_0) \rangle \qquad (3.15)$$

with $\boxed{i \partial_t \mathcal{U}(t, t_0) = H_{int}(t) \mathcal{U}(t, t_0)}$

Unitary time-evolution op. defines the S-matrix

$$S = \lim_{\substack{t_0 \to -\infty \\ t \to +\infty}} \mathcal{U}(t, t_0) \qquad (3.16)$$

Strictly speaking




λ is adiabatically switched on/off

$\Rightarrow$ $| \text{state } t \to -\infty \rangle = | i \rangle$ ← free

$| \text{state } t \to +\infty \rangle = | f \rangle$

Proper treatment: LSZ-formalism

Construction of $U(t,t_0)$:

Infinitesimal form of eq. (3.14), p. 41

$$|f(t+\Delta t)\rangle = |f(t)\rangle - i\Delta t H_{int}(t)|f(t)\rangle$$
$$= \left(1 - i\Delta t H_{int}(t)\right)|f(t)\rangle$$
$$= \left(1-i\Delta t H_{int}(t)\right)\left(1-i\Delta t H_{int}(t-\Delta t)\right)|f(t-\Delta t)\rangle \tag{3.17}$$

Iteration:

$$\Rightarrow |f(t+\Delta t)\rangle = \underbrace{\prod_{n=0}^{N}\left(1-i\Delta t H_{int}(t-n\Delta t)\right)}_{U(t+\Delta t,\, t-N\Delta t)}|f(t-N\Delta t)\rangle \tag{3.18}$$

Expansion in powers of $\Delta t$:

$$U(t+\Delta t, t-N\Delta t) = 1+(-i)\Delta t\sum_{n=0}^{N} H_{int}(t-n\Delta t)$$
$$+(-i)^2(\Delta t)^2\sum_{n<m} H_{int}(t-n\Delta t)H_{int}(t-m\Delta t)$$
$$+\ldots$$

$\Delta t \to 0$ with $N\Delta t = t-t_0$

$$\Rightarrow 1+(-i)\int_{t}^{t_0} dt' H_{int}(t')$$
$$+(-i)^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt'' H_{int}(t')H_{int}(t'')$$
$$+\ldots \tag{3.19}$$

Finally

$$\boxed{\mathcal{U}(t,t_0) = T\exp\left\{-i\int_{t_0}^{t} dt' H_{int}(t')\right\}} \tag{3.20}$$

with $T\,A(t)\,B(t') = A(t)B(t')\,\Theta(t-t')$

$$+ B(t')A(t)\,\Theta(t'-t)$$

For example

$$\frac{1}{2}T\int_{t_0}^{t} dt'\, H_{int}(t')\int_{t_0}^{t} dt''\, H_{int}(t'')$$

$$= \frac{1}{2}\left\{\int_{t_0}^{t} dt'\, H_{int}(t')\int_{t_0}^{t'} dt''\, H_{int}(t'')\right.$$

$$\left. + \int_{t_0}^{t} dt''\, H_{int}(t'')\int_{t_0}^{t''} dt'\, H_{int}(t')\right\}$$

$$= \int_{t_0}^{t} dt'\, H_{int}(t')\int_{t_0}^{t'} dt''\, H_{int}(t'') \tag{3.21}$$

Remark: $H_{int} = \int d^3x\, \Phi^4(x) \sim a^2 a^{+2}$

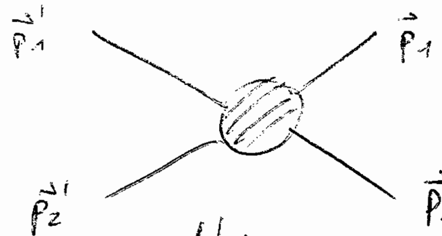
creates 2 particles and annihilates

them : $\langle 0|H_{int}|0\rangle$ infinite

vacuum processes

Example: 2 to 2 scattering

$\langle \vec{p}_1{}' \, \vec{p}_2{}' | \sim$ at $t \to +\infty$




$H_{int}$

$\sim | \vec{p}_1 \, p_2 \rangle$ at $t \to -\infty$

no scattering

S-matrix : $S = \mathbb{1} + iT$

$O(\lambda)$

In our case

$$iT_{fi} \underset{\substack{\uparrow \\ \text{infinities}}}{\simeq} -i \langle 0 | \, a(\vec{p}_1{}') \, a(\vec{p}_2{}') \, \lambda/_{4!} \int d^4x \, \phi^4(x) \, a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2) | 0 \rangle$$

$$\text{with} \quad \phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \left\{ a(\vec{p}) e^{-ipx} + a^\dagger(\vec{p}) e^{ipx} \right\} \tag{3.22}$$

We use $[a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3 \, 2\omega_{\vec{p}} \, \delta(\vec{p} - \vec{q})$ to pull the $a$'s in $H_{int}$ to the right: $\int d^4x \, \phi^4(x) \sim a^{\dagger 2} a^2$

$$\Rightarrow iT_{fi} \simeq \underbrace{-i\lambda}_{\text{Matrix element } \mathcal{M}} \Big[ \underbrace{(2\pi)^4 \, \delta^4(p_1 + p_2 - p_1' - p_2')}_{\text{energy-momentum conservation}} \Big] \tag{3.23}$$

$$iT_{fi} =: i\mathcal{M} \, (2\pi)^4 \, \delta^4(p_1 + p_2 - p_1' - p_2')$$

with $i\mathcal{M} = -i\lambda$

Remarks:

(i) Normal ordering:

$$:a(\vec{p}_1)\, a^\dagger(\vec{p}_2): \;=\; a^\dagger(\vec{p}_2)\, a(\vec{p}_1) \qquad (3.24)$$

e.g. Hamiltonian in free scalar theory eq. 2.62, p. 23:

$$:H_0: \;=\; \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\; :\frac{1}{2}a^\dagger(\vec{p})a(\vec{p}) + \frac{1}{2}a(\vec{p})a^\dagger(\vec{p}):$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\; a^\dagger(\vec{p})\, a(\vec{p}) \qquad (3.25)$$

no infinite vacuum terms

(ii) normal ordered Hint already gives eq. (3.23):

$$\lambda/4! \int d^4x :\phi^4(x): \;\sim\; \lambda/4! :a^{\dagger 2}a^2 + a^\dagger a a^\dagger a + a a^\dagger a a^\dagger$$

$$+ a^\dagger a^2 a^\dagger + a a^{\dagger 2} a + a^2 a^{\dagger 2}:$$

$$\sim \lambda/4\; a^{\dagger 2} a^2 \qquad (3.26)$$

(iii) Difference gives 'vacuum contributions'

$$H_{int} = :H_{int}: + \lambda/8 \int d^4x \left(\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\right)^2 \qquad (3.27)$$

$+ (a^\dagger a, a a^\dagger)$-terms $\quad [a, a^\dagger]^2 \quad \searrow\ \cap^2$

Computation of (3.23) : eq. (3.26)

$$\frac{\lambda}{4}\prod_i\int\frac{d^3q_i}{(2\pi)^3}\frac{1}{2\omega_{\vec{q}_i}}e^{-ix(q_3+q_4-q_2-q_1)}\langle 0|a(\vec{p}_1{}')a(\vec{p}_2{}')a^\dagger(\vec{q}_1)a^\dagger(\vec{q}_2)(\vec{q}_3)a(\vec{q}_4)a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)|$$

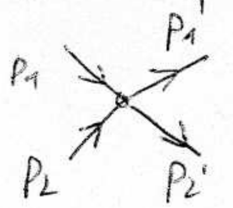
$$=4\frac{\lambda}{4}\prod_i\int\frac{d^3q_i}{(2\pi)^3}\frac{2\omega_{\vec{q}_i}}{2\omega_{\vec{q}_i}}\delta^3(\vec{p}_1{}'-\vec{q}_1)\,\delta^3(\vec{p}_2{}'-\vec{q}_2)\,\delta^3(\vec{p}_1-\vec{q}_3)\,\delta^3(\vec{p}_2-\vec{q}_4)\cdot e^{-ix(q_1+\dots+q_4}$$

$$=\lambda\, e^{-ix(p_1+p_2-p_1'-p_2')}$$

and

$$\int d^4x\; e^{-ix(p_1+p-p_1'-p_2')}=(2\pi)^4\,\delta^4(p_1+p_2-p_1'-p_2')$$

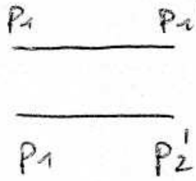
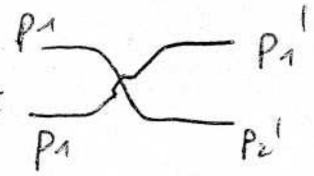
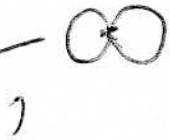
Interpretation:

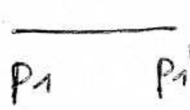
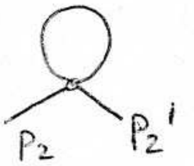



$: \; -i\lambda \quad \cdot \quad (2\pi)^4 \delta(p_1 + p_2 - p_1' - p_2')$

interaction strength    4-momentum conservation

vacuum parts: $\langle 0 | a(\vec{p}_1{}') \, a(\vec{p}_2{}') \, a^\dagger(\vec{p}_1) \, a^\dagger(\vec{p}_2) | 0 \rangle$

$-i\lambda \Big($  $+$  $\Big) \cdot \int d^4x \, O^2$ , 


$-i\lambda \Big[$  $\cdot$  $+ (p_1 \leftrightarrow p_2) + (p_1' \leftrightarrow p_2') + (p_1 \leftrightarrow p_2, p_1' \leftrightarrow p_2') \Big]$


loop contribution

First term:

$$\langle 0 | a(\vec{p}_1{}') \, a(\vec{p}_2{}') \, a^\dagger(\vec{p}_1) \, a^\dagger(\vec{p}_2) | 0 \rangle \Big( \underbrace{1 - i\lambda \int d^4x \, O^2}_{\exp\{-i\lambda \int d^4x \, O^2\} \; + O(\lambda^2)} \Big)$$

Phase / loops are infinite: call for appropriate treatment

Core example:

two-point fct.: propagator

(i) Start in Heisenberg picture:

vacuum of full theory: $|\Omega\rangle$

with $i\partial_t |\Omega\rangle = 0$ $\quad$ (3.28)

Operators evolve with full Hamiltonian, i.e.

$$i\partial_t \phi_H = [\phi_H, H]$$
$$\text{with } \phi_H = e^{-iHt}\phi(0,\vec{x})e^{iHt} \quad (3.29)$$

Link to interaction picture:

interaction picture states $|f\rangle_I$ evolve with $H_{int}$

$$|f(t)\rangle_I = \mathcal{U}(t,0)\,|f(0)\rangle_I \quad (3.30)$$

operators

$$i\partial_t \phi_I = [\phi_I, H_{int}] \quad (3.31)$$

It follows: (4th tutorial: $\mathcal{U}(t,0)=e^{iH_0t}e^{-iHt}$)

$$\phi_H(x)=\mathcal{U}(0,x_0)\,\phi_I(x)\,\mathcal{U}(x^0,0) \qquad (3.32$$

with $\phi_H(x)|f\rangle_H = \mathcal{U}(0,x^0)\,\phi_I(x)\,\mathcal{U}(x^0,0)|f\rangle_H$

and $i\partial_t\,\mathcal{U}(x^0,0)|f\rangle_H = H_{int}\,\mathcal{U}(x^0,0)|f\rangle_H$

It is tempting to identify $\mathcal{U}(x^0,0)|f\rangle_H$ with the interaction picture states $|f(t)\rangle_I$. At $t\to\pm\infty$, $\lambda$ is switched off adiabtically, and $|f\rangle_I$ tend to free in/out states. We have

$$\langle\Omega|\mathcal{U}(0,x^0)=\langle\Omega|\overset{\mathcal{U}(\infty,0)^{-1}}{\overset{\shortparallel}{\mathcal{U}(0,\infty)}}\,\mathcal{U}(\infty,x^0)$$

$$=\sum_n\langle\Omega|\mathcal{U}(0,\infty)|n\rangle_I\;{}_I\langle n|\mathcal{U}(-\infty,x^0)$$

adiabaticity: $|n\text{-part.}\rangle_{free}(t=-\infty)\xrightarrow{\mathcal{U}}|n\text{-particle}\rangle_{full}$

$$=\underbrace{\langle\Omega|\mathcal{U}(0,\infty)|0\rangle}_{\text{Please, see p. 51a}}\langle 0|\mathcal{U}(-\infty,x^0)$$

Also:

$$\mathcal{U}(x^0,0)|\Omega\rangle=\mathcal{U}(x^0,-\infty)|0\rangle\langle 0|\mathcal{U}(-\infty,0)|\Omega\rangle$$

Propagator:

$$\langle \Omega | T \phi_H(x)\, \phi_H(y) | \Omega \rangle$$

$$= \langle \Omega | \phi_H(x)\, \phi_H(y) | \Omega \rangle\, \Theta(x^0 - y^0)$$
$$+ \langle \Omega | \phi_H(y)\, \phi_H(x) | \Omega \rangle\, \Theta(y^0 - x^0) \qquad (3.35)$$

For $x^0 > 0 > y^0$:

$$\langle \Omega | \phi_H(x)\, \phi_H(y) | \Omega \rangle$$

$$(3.32) \longrightarrow \quad = \langle \Omega | \mathcal{U}(0,x^0)\, \phi_I(x)\, \overset{\mathcal{U}(x^0,0)\,\mathcal{U}(0,y^0)}{\overset{\shortparallel}{\mathcal{U}(x^0,y^0)}}\, \phi_I(y)\, \mathcal{U}(y^0,0) | \Omega \rangle$$

$$= \langle 0 | \mathcal{U}(\infty,x^0)\, \phi_I(x)\, \mathcal{U}(x^0,y^0)\, \phi_I(y)\, \mathcal{U}(y^0,-\infty) | 0 \rangle$$
$$1/\left( \langle \Omega | \mathcal{U}(0,\infty) | 0 \rangle \langle 0 | \mathcal{U}(-\infty,0) | \Omega \rangle \right)^{-1}$$
$$(3.36)$$

where we have used that in general

$$\mathcal{U}(x^0, y^0)\, \mathcal{U}(y^0, z^0) = \mathcal{U}(x^0, z^0) \qquad (3.37)$$

for $x^0 > y^0 > z^0$. This follows straightforwardly from eq. (3.20), p. 44.

Note that the denominator in (3.36) is (a product of two) phases, see p. 51a. We again use the adiabaticity, as in (3.34), and get

$$\langle\Omega|\mathcal{U}(0,\infty)|0\rangle^{-1}\langle 0|\mathcal{U}(-\infty,0)|\Omega\rangle^{-1}$$

p. 51a → $$= \langle 0|\mathcal{U}(\infty,0)|\Omega\rangle\langle\Omega|\mathcal{U}(0,-\infty)|0\rangle$$

(3.37) → $$= \langle 0|\mathcal{U}(-\infty,\infty)|0\rangle$$

$$= \langle 0|T\exp\left\{-i\int dt\, H_{int}(t)\right\}|0\rangle \qquad (3.38)$$

We also have for the numerator of (3.36)

$$\langle 0|\mathcal{U}(\infty,x^0)\,\phi_I(x)\,\mathcal{U}(x^0,y^0)\,\phi_I(y)\,\mathcal{U}(y^0,\infty)|0\rangle$$

$$= \langle 0|T\,\phi_I(x)\,\phi_I(y)\,\exp\left\{-i\int dt\, H_{int}(t)\right\}|0\rangle \qquad (3.37)$$

Finally, with $\phi_I=\phi$ and the analogous result for $y^0>x^0$;

$$\boxed{\langle\Omega|T\,\phi_H(x)\,\phi_H(y)|\Omega\rangle = \frac{\langle 0|T\,\phi(x)\,\phi(y)\,\exp\left\{-i\int dt\, H_{int}(t)\right\}|0\rangle}{\langle 0|T\exp\left\{-i\int dt\, H_{int}(t)\right\}|0\rangle}} \qquad (3.38)$$

$|\langle \Omega | \mathcal{U}(0,\infty) | 0 \rangle| = 1$ :

(i) $|\langle \Omega | \mathcal{U}(0, x^0)| = 1$ from $\mathcal{U}$ being unitary:

$$|\langle \Omega | \mathcal{U}(0, x^0)|^2 = \langle \Omega | \mathcal{U}(0,x^0)\,\mathcal{U}^\dagger(0,x^0)|\Omega\rangle$$

$$= \langle \Omega | \Omega \rangle = 1$$

(ii) $|\langle 0 | \mathcal{U}(-\infty, x^0)|^2 = 1$ from $\mathcal{U}$ being unitary

and $\langle 0 | 0 \rangle = 1$

From (i) and (ii) it follows that

$|\langle \Omega | \mathcal{U}(0,\infty) | 0 \rangle| = 1$. Hence

$$\langle \Omega | \mathcal{U}(0,\infty)|0\rangle^{-1} = \langle \Omega | \mathcal{U}(0,\infty)|0\rangle^{*}$$

$$= \langle 0 | \mathcal{U}^\dagger(0,\infty)|\Omega\rangle$$

Eq. (3.38) is straightforwardly extended to

$$\langle \Omega | T \phi_H(x_1) \cdots \phi_H(x_n) | \Omega \rangle$$

$$= \frac{\langle 0 | T\, \phi(x_1) \cdots \phi(x_n)\, e^{-i\int dt\, H_{int}} | 0 \rangle}{\langle 0 | T\, e^{-i\int dt\, H_{int}} | 0 \rangle} \qquad (3.39)$$

Remarks:

(i) The denominator in (3.38), (3.39) is a phase. For example, the linear term in $\lambda$ is

$$-i \langle 0 | \int dt\, H_{int} | 0 \rangle = -i\lambda \langle 0 | \int d^4x\, \phi^4 | 0 \rangle$$

$$= -i/8 \int d^4x \left( \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_{\vec{q}}} \right)^2 \qquad (3.40)$$

$\Rightarrow$ cancels vacuum term in (3.27)

(ii) The phase factor is infinite, as are the vacuum contributions in the numerator: the infinities cancel!