3.2. Wick's theorem

We have seen, that the computation of scattering amplitudes relates to the comp. of time-ordered n-point fcts

$$\langle 0 | T \phi(x_1) \cdots \phi(x_n) e^{i\int d^4x \mathcal{L}_{int}} | 0 \rangle$$

where $-\int dt\, H_{int} = \int dt\, L_{int} = \int d^4x\, \mathcal{L}_{int}$, and the coupling $\lambda \ll 1$. Since

$$\langle 0 | T\, \phi(x_1) \cdots \phi(x_n) \left(\mathcal{L}_{int}\right)^m | 0 \rangle$$

$$= \langle 0 | T\, \phi(x_1) \cdots \phi(x_{n+4m}) | 0 \rangle$$

with $x_{n+1} = \cdots = x_{n+4m} = x$, the only building block is

$$\boxed{\langle 0 | T\, \phi(x_1) \cdots \phi(x_n) | 0 \rangle} \qquad (3.41)$$

If $x_1^0 > x_2^0 > \cdots > x_n^0$, this is

$$\langle 0 | \phi(x_1) \cdots \phi(x_n) | 0 \rangle$$

and we simply have to use the canonical commutation relations for $a, a^\dagger$ (Note that $\phi = \phi_I$ is free!)

Strategy: write $T\phi(x_1)\cdots\phi(x_n)$ as $:\phi(x_1)\cdots\phi(x_n): +$ rest. Taking the vacuum expectation value, the normal ordered part vanishes!

contains $:\phi(x_1)\cdots\phi(x_{n-2}) \cdots$

Example: two-point fct.

$$\phi(x) = \phi_+(x) + \phi_-(x)$$

with $\phi_+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec p}} a^+(\vec p) e^{ipx}$

creation

annihilation $\longrightarrow$ $\phi_-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec p}} a(\vec p) e^{-ipx}$

$$(3.42)$$

For $x^0 > y^0$:

$$T\,\phi(x)\phi(y) = \phi_+(x)\phi_+(y) + \phi_+(x)\phi_-(y) + \phi_-(x)\phi_+(y) + \phi_-(x)\phi_-(y)$$

$$= \phi_+(x)\phi_+(y) + \phi_+(x)\phi_-(y) + \phi_+(y)\phi_-(x) + \phi_-(x)\phi_-(y)$$

$$+ [\phi_-(x), \phi_+(y)]$$

$$(3.43)$$

$$\Rightarrow \; T\,\phi(x)\,\phi(y)\Big|_{x_0>y_0} = \;:\phi(x)\,\phi(y): + [\phi_-(x), \phi_+(y)] \qquad (3.44)$$

where $:\phi_-(x)\,\phi_+(y): = \phi_+(y)\,\phi_-(x) \quad \forall x$

from $:a(\vec{p})\,a^+(\vec{q}): = a^+(\vec{q})\,a(\vec{p})$ (3.45)

Taking vacuum expectation values, the normal ordered part vanishes:

$$\boxed{\langle 0|\,T\phi(x)\,\phi(y)\,|0\rangle = [\phi_-(x), \phi_+(y)]\,\Theta(x^0-y^0) + [\phi_-(y), \phi_+(x)]\,\Theta(y^0-x^0)} \qquad (3.46)$$

The time-ordered propagator is called Feynman propagator:

$$D_F(x-y) = \langle 0|\,T\phi(x)\,\phi(y)\,|0\rangle \qquad (3.47)$$

It is the key-ingredient in (time-ordered) perturbation theory.

Computation of D_F:

$$[\phi_-(x),\ \phi_+(y)]\ \Theta(x^0-y^0)$$

$$= \int \frac{d^3q}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec p}}\frac{1}{2\omega_{\vec q}}\left[a(\vec p), a^\dagger(\vec q)\right] e^{-ipx+iqy}\,\Theta(x^0-y^0)$$

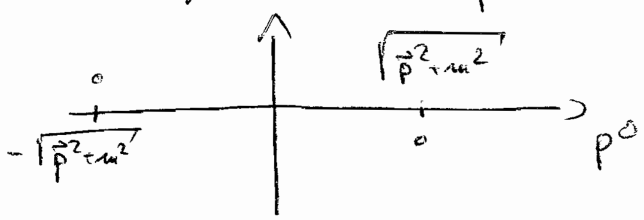
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec p}}\ e^{-ip(x-y)}\ \Theta(x^0-y^0)$$

$$\Rightarrow\quad D_F(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec p}}\left\{e^{-ip(x-y)}\Theta(x^0-y^0) + e^{ip(x-y)}\Theta(y^0-x^0)\right\} \tag{3.48}$$

D_F can be rewritten as $(\varepsilon \to 0_+)$

$$\boxed{D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}\,e^{-ip(x-y)} \qquad (3.49)}$$

Proof: The integrand has poles at $p^0 = \pm\sqrt{\vec p^{\,2}+m^2-i\varepsilon}$



$x^0-y^0>0$: close contour in lower half plane
$x^0-y^0<0$: " " " upper " "

$x^0 - y^0 > 0$: pole at $p_-^0 = \sqrt{\vec{p}^2 + m^2 - i\varepsilon} \to \omega_{\vec{p}}$

$$D_F(x-y) = -\int \frac{d^3p}{(2\pi)^3} \frac{2\pi i}{2\pi} \operatorname*{res}_{p_-^0} \left[\frac{e^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \, i \, \frac{e^{-ip(x-y)}}{2i\omega_{\vec{p}}} \Bigg|_{p^0 = \omega_{\vec{p}}} \qquad (3.48)$$

Similarly for $x^0 - y^0 < 0$. $\square$

Remark:

(i) We have parameterised the time-ordered propogator in terms of commutators. On operator level we have

$$T\,\phi(x)\phi(y) = :\phi(x)\phi(y): \qquad (3.49)$$
$$+ [\phi_-(x), \phi_+(y)]\,\Theta(x^0 - y^0)$$
$$+ [\phi_-(y), \phi_+(x)]\,\Theta(y^0 - x^0)$$

or
$$T\,\phi(x)\,\phi(y) = :\phi(x)\,\phi(y): + \underbrace{\overbracket{\phi(x)\,\phi(y)}}_{\text{contraction}} \qquad (3.50)$$

where
$$\overbracket{\phi(x)\,\phi(y)} = [\phi_-(x), \phi_+(y)]\,\Theta(x^0-y^0) + [\phi_-(y), \phi_+(x)]\,\Theta(y^0-x^0) \qquad (3.51)$$
$$= D_F(x-y) \longleftarrow \text{c-number}$$

Generalisation to product of n fields:

Wick's Theorem:

$$\boxed{\begin{aligned} &T\,\phi(x_1)\cdots\phi(x_n) \\ &\qquad = :\phi(x_1)\cdots\phi(x_n) + \text{all contractions}: \end{aligned}} \qquad (3.52)$$

where
$$\phi(x_1)\cdots\overbracket{\phi(x_i)\cdots\phi(x_j)}\cdots\phi(x_n)$$
$$= \phi(x_1)\cdots\phi(x_{i-1})\,\phi(x_{i+1})\cdots\phi(x_{j-1})\,\phi(x_{j+1})\cdots\phi(x_n)\;\overbracket{\phi(x_i)\,\phi(x_j)}$$

Example:

(1) $$T\,\phi(x_1)\,\phi(x_2) = :\phi(x_1)\,\phi(x_2) + \wick{\c\phi(x_1)\,\c\phi(x_2)}$$

(2) 4-point correlation function

$$T\,\phi(x_1)\cdots\phi(x_4) = T\,\phi_1\phi_2\phi_3\phi_4$$

$$\begin{aligned}
= {}& :\phi_1\phi_2\phi_3\phi_4 + \wick{\c\phi_1\c\phi_2}\phi_3\phi_4 + \phi_1\phi_2\wick{\c\phi_3\c\phi_4} + \wick{\c\phi_1\phi_2\c\phi_3}\phi_4 \\
& + \wick{\c\phi_1\phi_2\phi_3\c\phi_4} + \phi_1\wick{\c\phi_2\c\phi_3}\phi_4 + \phi_1\wick{\c\phi_2\phi_3\c\phi_4} \\
& + \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4} :
\end{aligned}$$

(5.53)

where e.g.

$$:\wick{\c\phi_1\c\phi_2}\phi_3\phi_4: = :\phi_3\phi_4:\,\wick{\c\phi_1\c\phi_2} = :\phi_3\phi_4:\,D_F(x_1-x_2)$$

(5.54)

Note also that

$$\langle 0|:\mathcal{O}:|0\rangle = 0 \qquad (5.55)$$

It follows with eq. (3.53) and eq. (3.55) that

$$\langle 0|\,T\,\phi(x_1)\cdots\phi(x_4)\,|0\rangle \qquad (3.56)$$

$$\begin{aligned}
&= D_F(x_1-x_2)\,D_F(x_2-x_3) + D_F(x_1-x_3)\,D_F(x_2-x_4) \\
&\quad + D_F(x_1-x_4)\,D_F(x_2-x_3)
\end{aligned}$$

Proof of Wick theorem (by induction)

(i) $n=1,2$: $T\phi_1 = :\phi_1:$ ✓

$$T\phi_1\phi_2 = :\phi_1\phi_2: + \overbracket{\phi_1\phi_2}$$ ✓

(ii) Assume, Wick's theorem applies to

$T\phi_2 \cdots \phi_{n+1}$. Without loss of gen.: $\boxed{x_1^0 \geq x_i^0 \ \forall i}$
(time-ordering)

Then $T\phi_1 \cdots \phi_{n+1}$

$$= \phi_1 (:\phi_2 \cdots \phi_{n+1} + \text{all contractions}:)$$

$$= (\phi_{1+} + \phi_{1-})(:\phi_2 \cdots \phi_{n+1} + \text{all contr.}:)$$

$$= :\phi_1 \cdots \phi_{n+1} + [\phi_{1-}, \phi_2]\phi_3 \cdots \phi_{n+1}$$

$$+ \phi_2 [\phi_{1-}, \phi_3]\phi_4 \cdots \phi_{n+1} + \cdots \phi_2 \cdots [\phi_{1-}, \phi_{n+1}]:$$

$$+ (\phi_{1+} + \phi_{1-})(:\text{all contr.}:) \qquad (3.57)$$

We use

$$[\phi_{1-}, \phi_i] = [\phi_{1-}, \phi_{i+}] = \overbracket{\phi_1 \phi_i}$$

and the same identities in (3.57) for

$(\phi_{1+} + \phi_{1-})(:\text{all contr.}:)$

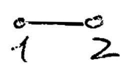
Then

$$T\,\phi_1 \cdots \phi_{n+1} = :\phi_1 \cdots \phi_{n+1} + \text{all contr.}:$$

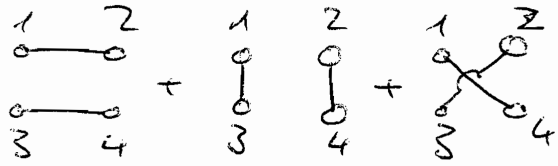
∎

3.3 Feynman rules

With Wick's theorem we write every time-ordered n-point fct. as a product of Feynman propagators. We introduce the diagrammatical notation

$$D_F(x_1 - x_2) = \langle 0 | T\,\phi_1\,\phi_2 | 0 \rangle =$$ 

It follows e.g.:

$$\langle 0 | T\,\phi_1 \cdots \phi_4 | 0 \rangle =$$ 

What about $\langle 0 | T\,\phi_1\,\phi_1 | 0 \rangle$?

$$D_F(0) =$$