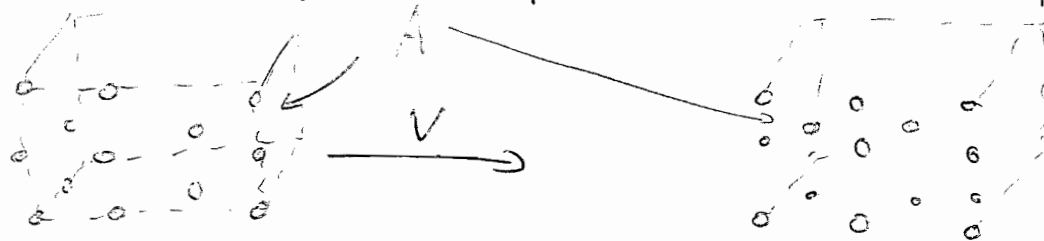


3.4. Cross Section

Fixed target experiment: Example



bundle of particles

type B: bundle length l_B

density ρ_B

velocity v

bundle of particles

type A: bundle length l_A ,

density ρ_A

velocity 0

cross section σ :

$$\sigma = \frac{\text{Number of events} = N_{\text{events}}}{(N_B N_A)/A} \quad (3.66)$$

where

A scattering area (transverse)

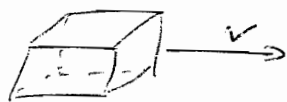
Space-dependent density:

$$(N_B N_A) / A = \int_A d^2x \rho_A(x) \rho_B(x) l_A l_B \quad (3.67)$$

$$\Rightarrow \sigma = \frac{N_{\text{events}}}{l_B \int_A d^2x \rho_A(x) \rho_B(x)}$$

$$\text{const. densities} \rightarrow = \frac{N_{\text{events}}}{l_A \rho_A l_B \rho_B \cdot A} \quad (3.68)$$

For the above example, we need to consider states, that are localized in

space/mom.: 

Wave packet:

$$|f_{\vec{p}}\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} f_{\vec{p}}(\vec{k}) |\vec{k}\rangle \quad (3.69)$$

with $f_{\vec{p}}(\vec{k})$ is peaked at \vec{p}

e.g. $f_{\vec{p}}(\vec{k}) \sim e^{-(\vec{k}-\vec{p})^2/\sigma^2}$ Gaussian

Normalisation: $\langle f_{\vec{p}} | f_{\vec{p}} \rangle = 1$

$$= \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \frac{1}{2\omega_{\vec{k}'}} f_{\vec{p}}^*(\vec{k}') f_{\vec{p}}(\vec{k}) \langle \vec{k}' | \vec{k} \rangle$$

$$= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} |f_{\vec{p}}|^2(\vec{k}) \quad (3.70)$$

Gaussian is localised in \vec{k} and \vec{x} : a

Fourier transform of a Gaussian is a Gaussian

In operator language,

$$|f_{\vec{p}}\rangle = \int d^3 x \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2} e^{i\vec{k}\vec{x}} f(\vec{k}) \phi(\vec{x}) |0\rangle$$

$$= i \int d^3 x \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{i\vec{k}\vec{x}} f(\vec{k}) \pi(\vec{x}) |0\rangle$$

(3.71)

from

$$a^\dagger(\vec{k}) = \int d^3 x e^{i\vec{k}\vec{x}} \left\{ \omega_{\vec{k}} \phi(\vec{x}) - i\bar{\pi}(\vec{x}) \right\}$$

(3.72)

see p. 19-20

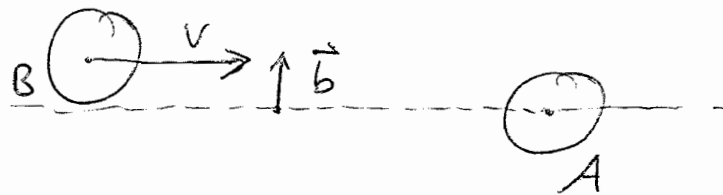
Remark: probab. for 1 part.: $\int \frac{d^3 q}{(2\pi)^3} |\langle \vec{q} | f_{\vec{p}} \rangle|^2$

$$= \int \frac{d^3 q}{(2\pi)^3} |f_{\vec{p}}(\vec{q})|^2 = 1$$

Initial state in our case: $|\vec{k}_A, \vec{k}_B\rangle = a^\dagger(\vec{k}_A) a^\dagger(\vec{k}_B) |0\rangle$

$$|i\rangle = \int \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}_A}} \frac{1}{2\omega_{\vec{k}_B}} f_{\vec{p}_A}(\vec{k}_A) f_{\vec{p}_B}(\vec{k}_B) |\vec{k}_A, \vec{k}_B\rangle \quad (3.73)$$

Impact parameter \vec{b} :



Translation is generated by momentum operator \vec{P} : (p. 31 eqs. (2.90), (2.91))

$$e^{-i\vec{P}\vec{b}} |\vec{k}\rangle = e^{-i\vec{k}\vec{b}} |\vec{k}\rangle \quad (3.74)$$

$$\Rightarrow e^{-i\vec{P}\vec{b}} |f_{\vec{p}}\rangle$$

$$= \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{i\vec{k}(\vec{x}-\vec{b})}$$

$$\cdot (\phi(\vec{x}) \omega_{\vec{k}} - i\pi(x)) |0\rangle$$

$$(3.75)$$

\Rightarrow Initial state with impact parameter \vec{b} :

$$|i_b^{\pm}\rangle = \int \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}_A}} \frac{1}{2\omega_{\vec{k}_B}} f_{\vec{p}_A}^{\pm}(\vec{k}_A) f_{\vec{p}_B}^{\pm}(\vec{k}_B) \cdot e^{-i\vec{k}_B \cdot \vec{b}} |\vec{k}_A, \vec{k}_B\rangle \quad (3.76)$$

Transition amplitude:

$$\langle \vec{p}_1, \vec{p}_2 | S | i_b^{\pm} \rangle$$

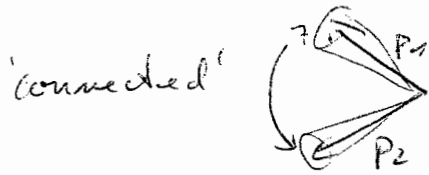
with probability $|\langle \vec{p}_1, \vec{p}_2 | S | i_b^{\pm} \rangle|^2$. The number of events in a dense beam is (single target part.)

$$N_{\text{events}} = \frac{N_B}{A} \int_A d^2b |\langle \vec{p}_1, \vec{p}_2 | S | i_b^{\pm} \rangle|^2 \quad (3.77)$$

with cross section

$$\sigma(\vec{p}_1, \vec{p}_2) = \int d^2b |\langle \vec{p}_1, \vec{p}_2 | S | i_b^{\pm} \rangle|^2 \quad (3.78)$$

More realistic is a detection of a moment in region V_f , e.g.



$$\Rightarrow \sigma(V_f) = \int_{V_f} \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}_1}} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}_2}} \int d^2 b |\langle \vec{p}_1, \vec{p}_2 | S | i_b \rangle|^2$$

$$\underbrace{\frac{d^4 p_1}{(2\pi)^4} 2\pi \delta(p_1^2 - m^2)}_{\leftarrow \text{on-shell}} \quad (3.79)$$

or, differentially and for n particles

$$d\sigma = \frac{1}{n!} \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}_i}} \int d^2 b |\langle \vec{p}_1, \dots, \vec{p}_n | S | i_b \rangle|^2 \quad (3.80)$$

Assume now that the \vec{p}_i are not parallel to \vec{p}_B (no (trivial) forward scattering).

We use that

$$S_{fi} = \mathbb{1}_{fi} + iT_{fi}, \quad iT_{fi} = i\mathcal{M}_{fi} (2\pi)^4 \delta^4(\sum \vec{p}_f - \sum \vec{k}_i)$$

and conclude

$$\begin{aligned}
 d\sigma &= \frac{1}{i} \frac{d^3 p_{fi}}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}_i}} \int d^2 b \int \frac{d^3 k_A}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}_A}} \frac{d^3 k_B}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}_B}} \\
 &\cdot \int \frac{d^3 k'_A}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}'_A}} \frac{d^3 k'_B}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}'_B}} f_{\vec{p}_A}(\vec{k}_A) f_{\vec{p}_B}(\vec{k}_B) f_{\vec{p}_A}^*(\vec{k}'_A) f_{\vec{p}_B}^*(\vec{k}'_B) \\
 &\cdot e^{i\vec{b}(\vec{k}'_B - \vec{k}_B)} |M_{fi}|^2 (2\pi)^4 \delta^4(\sum p_{fi} - \sum k_i) \\
 &\cdot (2\pi)^4 \delta^4(\sum p_{fi} - \sum k'_i) \quad (3.81)
 \end{aligned}$$

with $k_1 = k_A$, $k_2 = k_B$.

Computations:

$$(i) \int d^2 b e^{i\vec{b}(\vec{k}'_B - \vec{k}_B)} = (2\pi)^2 \delta^2(k'_{B\perp} - k_{B\perp}) \quad (3.82)$$

$\nearrow k_{\perp} = (k_1^{\perp}, k_2^{\perp})$
 \nwarrow

$$\begin{aligned}
 (ii) &\int d^3 k'_A d^3 k'_B \delta^4(\sum p_{fi} - \sum k_i) \delta^2(k'_{B\perp} - k_{B\perp}) \\
 &= \int d^3 k_A^{3'} d^3 k_B^{3'} \delta(\sum p_{fi}^3 - \sum k_i^3) \delta(\sum p_{fi}^0 - \sum k_i^0) \\
 &= \int d^3 k_A' \delta(\sum p_{fi}^0 - \sum k_i^0) \quad (3.83)
 \end{aligned}$$

with $k_{B\perp}' = k_{B\perp}$, $k_{A\perp}' = k_{A\perp}$, $k_B^{3'} = \sum p_{fi}^3 - k_A^{3'}$

$$\Rightarrow \int d^3 k'_A d^3 k'_B \delta^4(\sum p_{fi} - \sum k_i') \delta^2(k'_{BL} - k_{BL})$$

$$= \int d^3 k'_A \delta\left(\sum p_{fi} - \sqrt{k_A'^2 + m_A^2} - \sqrt{k_B'^2 + m_B^2}\right) \left. \begin{array}{l} k_{A/BL} = k_{A/BL} \\ \sum k_i^{3i} = \sum k_i^{3i} \\ \sum k_i^{0i} = \sum k_i^{0i} \end{array} \right\}$$

$\nearrow k_A^2 + (k_A^{3i})^2$
 $\nearrow k_B^2 + (\sum p_i^{3i} - k_A^{3i})^2$

$$= \frac{1}{|k_A^{3i}/k_A^{0i} - k_B^{3i}/k_B^{0i}|} \xrightarrow{k_{A/B} = p_{A/B}} \frac{1}{|v_A - v_B|} \quad (3.84)$$

(iii) The wave packets $f_{\vec{p}_{A/B}}$ are localised about $\vec{p}_{A/B}$, hence we can substitute in all prefactors $k_{A/B} \rightarrow p_{A/B}$.

It follows

$$d\sigma = \frac{\pi}{i} \frac{d^3 p_{fi}}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}_{fi}}} \frac{1}{4 p_A^0 p_B^0 |v_A - v_B|}$$

$$\cdot \int \frac{d^3 k_A}{(2\pi)^3} \frac{d^3 k_B}{(2\pi)^3} \frac{1}{2k_A^0 2k_B^0} |f_{\vec{p}_A}(k_A)|^2 |f_{\vec{p}_B}(k_B)|^2$$

$$= |M_{fi}|^2 \cdot (2\pi)^4 \delta^4(\sum p_{fi} - \sum k_i) \quad (3.85)$$

Again we use the localisation to replace

$\sum k_i \rightarrow p_i = p_A + p_B$. We also use $\int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} |f_p(k)|^2 = 1$

$$\Rightarrow d\sigma = \frac{1}{4 p_A^0 p_B^0 |v_A - v_B|} |M_{fi}|^2 (2\pi)^4 \delta^4(P_f - P_i) \prod_{i=1}^n \frac{d^3 p_{fi}}{(2\pi)^3} \frac{1}{2p_{fi}^0} \quad (3.86)$$

invariant under
boosts along
beam axis (x^3)

n-particle phase space
 $=: d\pi_n$

Exercise \rightarrow

$d\sigma$ is a (differential) transverse area

Highly relativistic case:

$$S = (p_A + p_B)^2 \gg m_A^2, m_B^2$$

$$\Rightarrow 4 p_A^0 p_B^0 |v_A - v_B| \rightarrow 2S \leftarrow \text{show with}$$

boost in CMS

(center of Mass)

and

$$d\sigma = \boxed{\frac{1}{2S}} |M_{fi}|^2 d\pi_n$$

(3.87)

Eq. (3.87) can be rewritten in a manifestly boost-invariant way:

$$d\sigma = \frac{1}{2W(s, m_A^2, m_B^2)} \cdot |M_{fi}|^2 d\mathbb{T}_n \quad (3.88)$$

with $\omega(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2xy - 2xz - 2yz}$ Exercise

Example: 2-2 scattering in ϕ^4 -theory
in highly relativistic case

Phase space: $n=2$ in eq. (3.87), $p_{fi} = p_i$

$$\int d\mathbb{T}_2 = \int (2\bar{\alpha})^4 \delta^4(p_1 + p_2 - (p_A + p_B)) \cdot \frac{d^3 p_1}{(2\bar{\alpha})^3} \frac{1}{2p_1^0} \frac{d^3 p_2}{(2\bar{\alpha})^3} \frac{1}{2p_2^0} \quad (3.89)$$

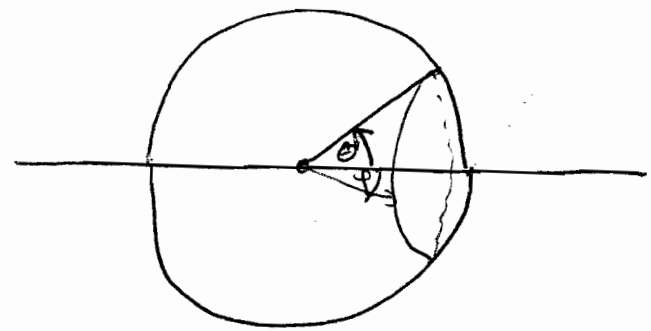
$$(p_A + p_B)^2 \gg m_A^2, m_B^2 \rightarrow \simeq \frac{1}{(2\bar{\alpha})^2 4p_1^0 p_2^0} \int d^3 p_2 \delta(p_1^0 + p_2^0 - \sqrt{s'})$$

with comp. in CMS: $\vec{p}_1 = -\vec{p}_2 \Rightarrow p_1^0 = p_2^0$

We also use

$$d^3 p_2 = d\Omega |\vec{p}_2|^2 d|\vec{p}_2| \quad (3.90)$$

with $d\Omega = d\varphi \sin\theta d\theta$



It follows, $p_1^0 + p_2^0 - \sqrt{s} \approx 2p_2^0 - \sqrt{s} = 2|\vec{p}_2| - \sqrt{s}$

$$\int d\pi_2 = \frac{1}{2} \frac{1}{(2\pi)^2 4 p_1^0 p_2^0} d\Omega \quad S/4 \quad (3.91)$$

Now we use (see p. 45, eq. 3.23), that

$$|M_{fi}|^2 = \lambda^2 \quad (3.92)$$

and hence

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{2s} |M_{fi}|^2 \int_{d\Omega(k) \text{ fixed}} d\pi_2 = \frac{\lambda^2}{64\pi^2 s}} \quad (3.93)$$

Remarks: (i) dim. counting: λ^2/s , (ii) angle-indep.: spin 0

Finally: Computation of S-Matrix elements

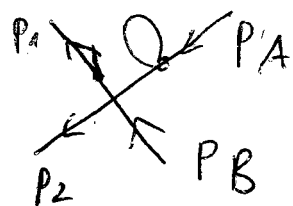
In the 2-2 scattering example we have used, that

$$|M_{fi}|^2 = \lambda^2 + \mathcal{O}(\lambda^3) \quad (3.94)$$

Expansion in Feynman diagrams:

$$\langle \vec{p}_1 \vec{p}_2 | iT | \vec{p}_A \vec{p}_B \rangle = \text{X} + (\text{X} + \text{perms}) + \text{X} + \dots \quad (3.95)$$

Compute:



$$\text{with } \frac{1}{2} \text{ loop} = -i\pi$$

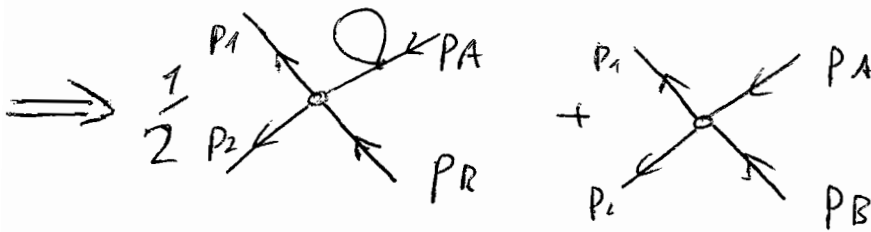
$$= \frac{i}{p_A^2 - m^2 - i\epsilon} \overset{\mathcal{O}(\lambda)}{\downarrow} (-i\pi) (-i\lambda)$$

$$\cdot (2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2)$$

$$(3.96)$$

$$= \left(\frac{i}{p_1^2 - m^2 + i\epsilon} \right)^{-1} \frac{i}{(p_A^2 - m^2 + i\epsilon)} (-i\bar{W}) \frac{i}{(p^2 - m^2 + i\epsilon)} (-i\lambda)^2$$

$$\cdot \delta^4(p_A + p_B - p_1 - p_2)$$



$$= -i\lambda \left(\frac{i}{p_1^2 - m^2 + i\epsilon} \right)^{-1} \left[\frac{i}{p_A^2 - m^2 + i\epsilon} + \frac{i}{p_A^2 - m^2 + i\epsilon} (-i\bar{W}) \frac{i}{p_A^2 - m^2 + i\epsilon} \right]$$

$$\cdot \delta^4(p_A + p_B - p_1 - p_2)$$

$$= -i\lambda \left(\frac{i}{p_A^2 - m^2 + i\epsilon} \right)^{-1} \frac{i}{p_A - (m^2 + \bar{W}) + i\epsilon} \delta^4(p_A + p_B - p_1 - p_2)$$

$$= \left[-i\lambda \delta^4(p_A + p_B - p_1 - p_2) \right] \xrightarrow{PA} \overset{-1}{\circ} \rightarrow \text{shaded circle} \xrightarrow{PA} \quad (3.97)$$

(bare) free inverse
propagation with p_A

full propagation
with p_A

Remarks:

(i) $\rightarrow_{P_A}^{-1}$ related to the fact,

that the particle A in the initial state was prepared as a free state, only true for $t \rightarrow -\infty$. The correct in-state should relate to full (inverse) propagation

(ii) $\rightarrow_{P_A}^{-1} \rightarrow \textcircled{\rightarrow}_{P_A}^{-1}$ leads to

$$1 = \left(\textcircled{\rightarrow}_{P_A} \right)^{-1} \left(\textcircled{\rightarrow}_{P_A} \right) \text{ in eq. (3.97):}$$

M_{fi} is computed by computing amputated, connected scattering diagrams.